Stanford Continuing Studies, Course MATH 07, Fall 2005

Three views of mathematics

LECTURE 2

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Perspectives on mathematics

- To the person who seeks to understand the world (inc. scientists):
  - **language** to describe and quantify the world
- To the person who wants to design/build/construct something (inc. engineers, moviemakers, etc.):
  - **language** to specify
  - **tools** to aid the process (inc. problem solving)
- To the person who wants to engage in trade and commerce:
  - precise tracking and planning **tool**
- To the mathematician:
  - a rich **field of study** having intrinsic aesthetic appeal

The mathematical method

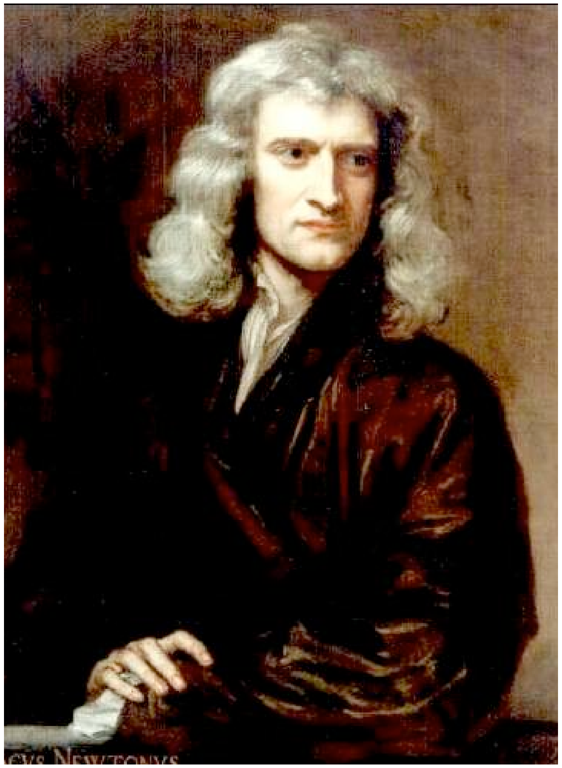
Mathematics is the science of patterns.

- Identify a particular pattern in the world.
- Study it.
- Develop a notation to describe it.
- Use that notation to further the study.
- Formulate basic assumptions (axioms) to capture the fundamental properties of the abstracted pattern.
- Study the abstracted pattern, establishing truth by means of rigorous proofs from the axioms.
- Develop procedures that you and others may use to apply the results of the study to the world.
- Apply the results to the world.

An example: CALCULUS

Infinitesimal calculus

One of the most successful and far reaching technologies of all time.



Isaac Newton

Gottfried Leibniz

Infinitesimal calculus

Motivating real world problem:

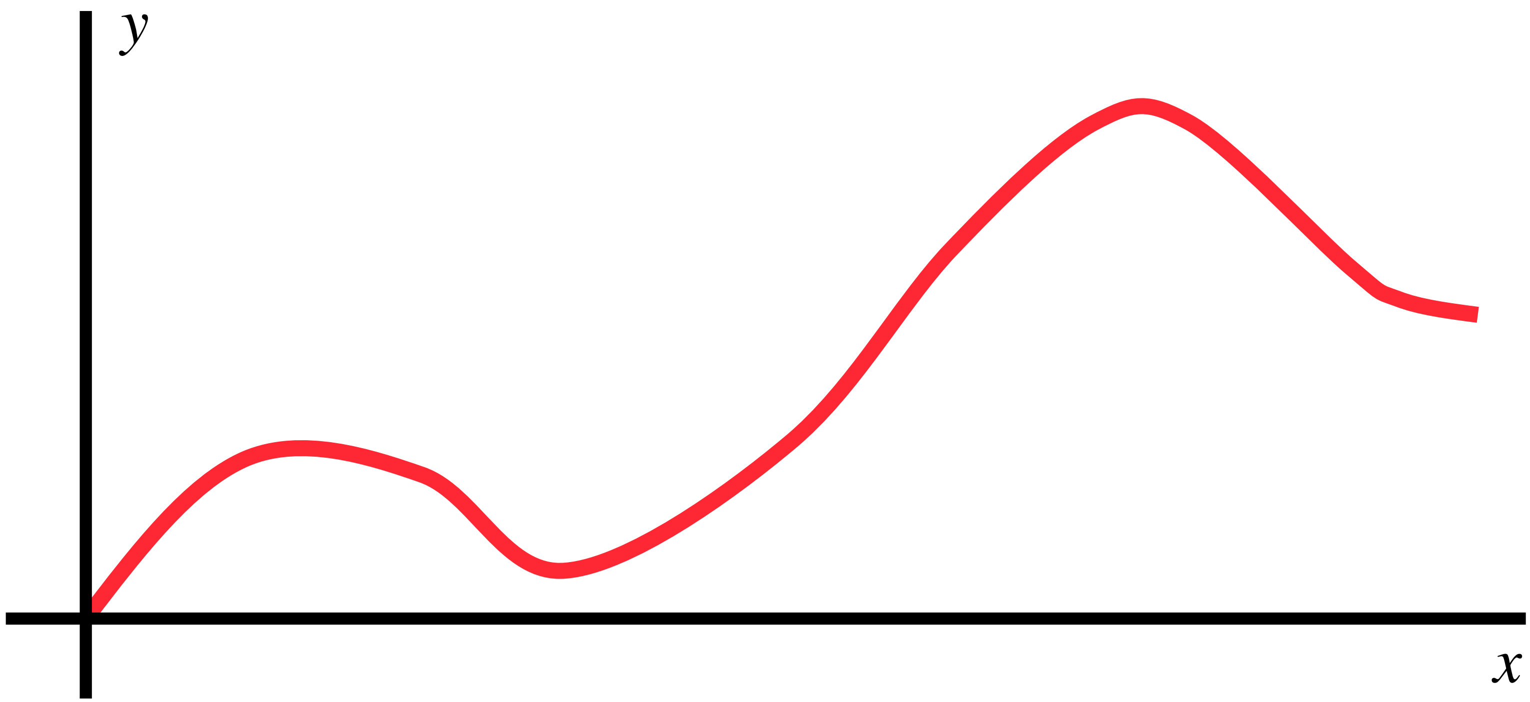
Understand and analyze precisely continuous motion and change.

Differential calculus

A method for calculating rates of change

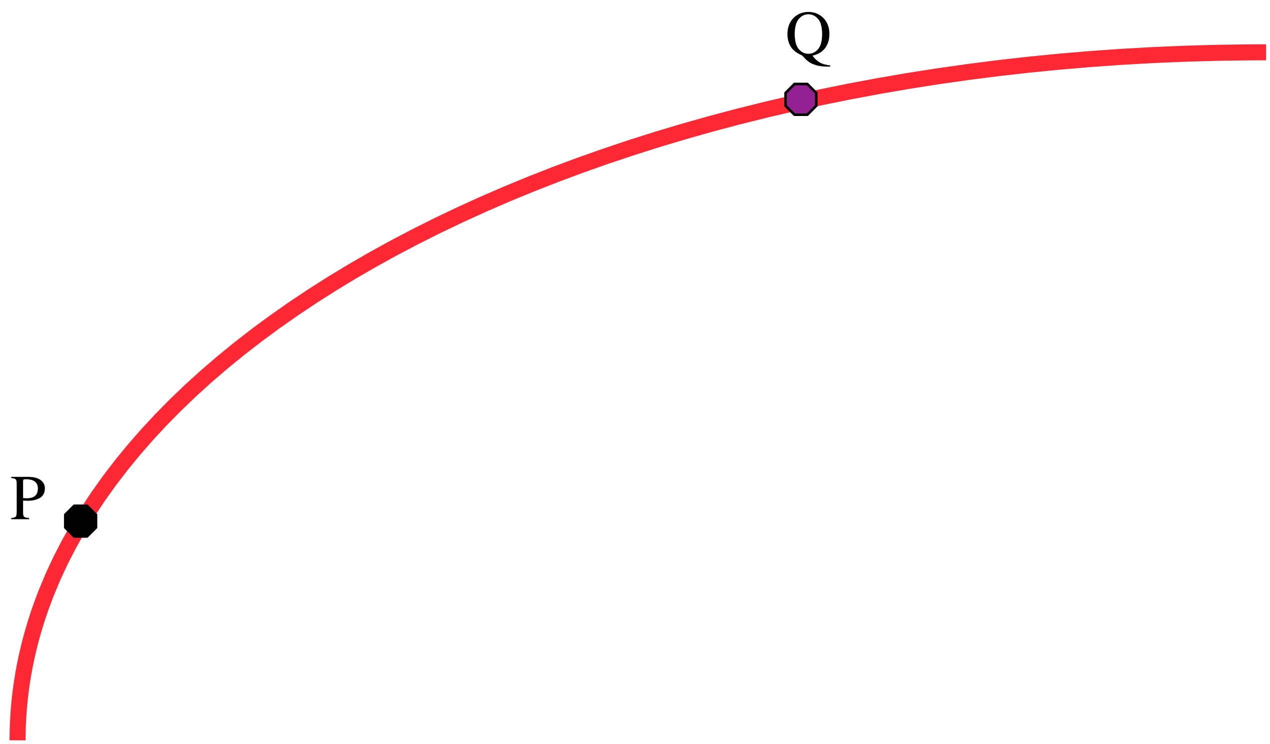
Differential calculus

Problem: To compute the slope of a curve at a given point.



Calculating the slope at a point

Calculating the slope at a point



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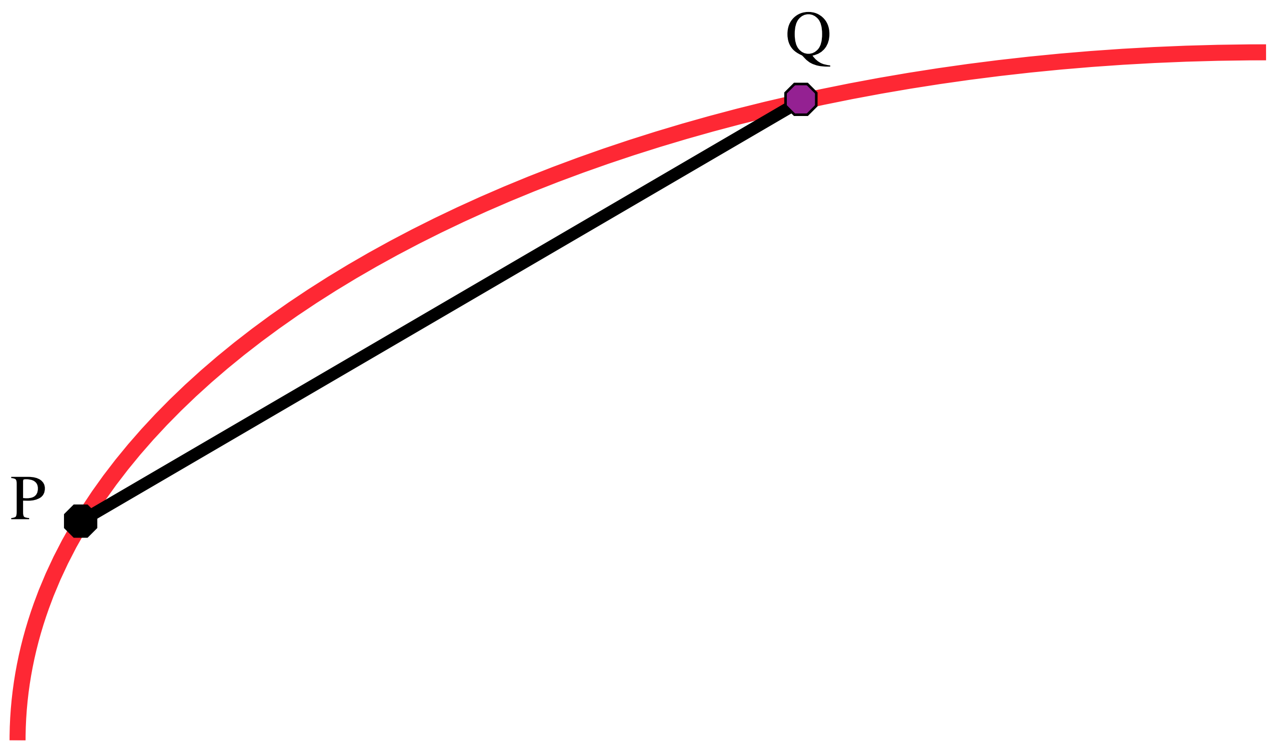

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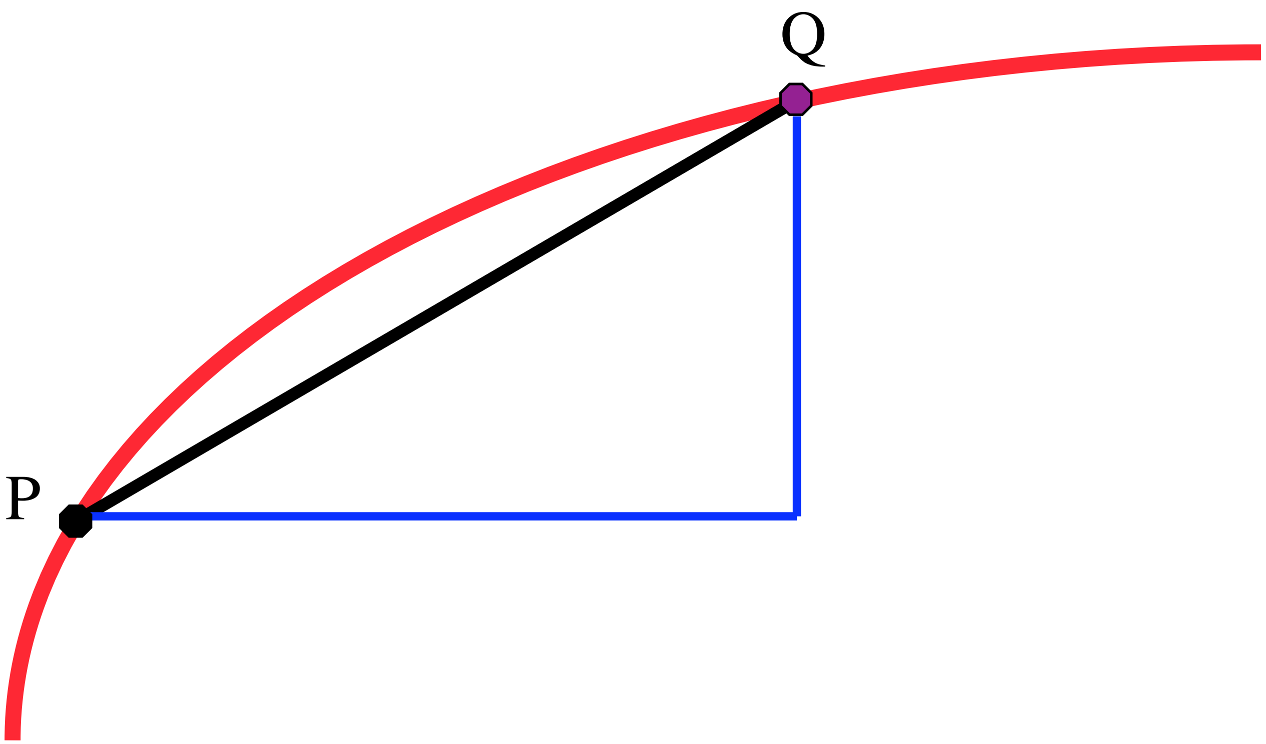

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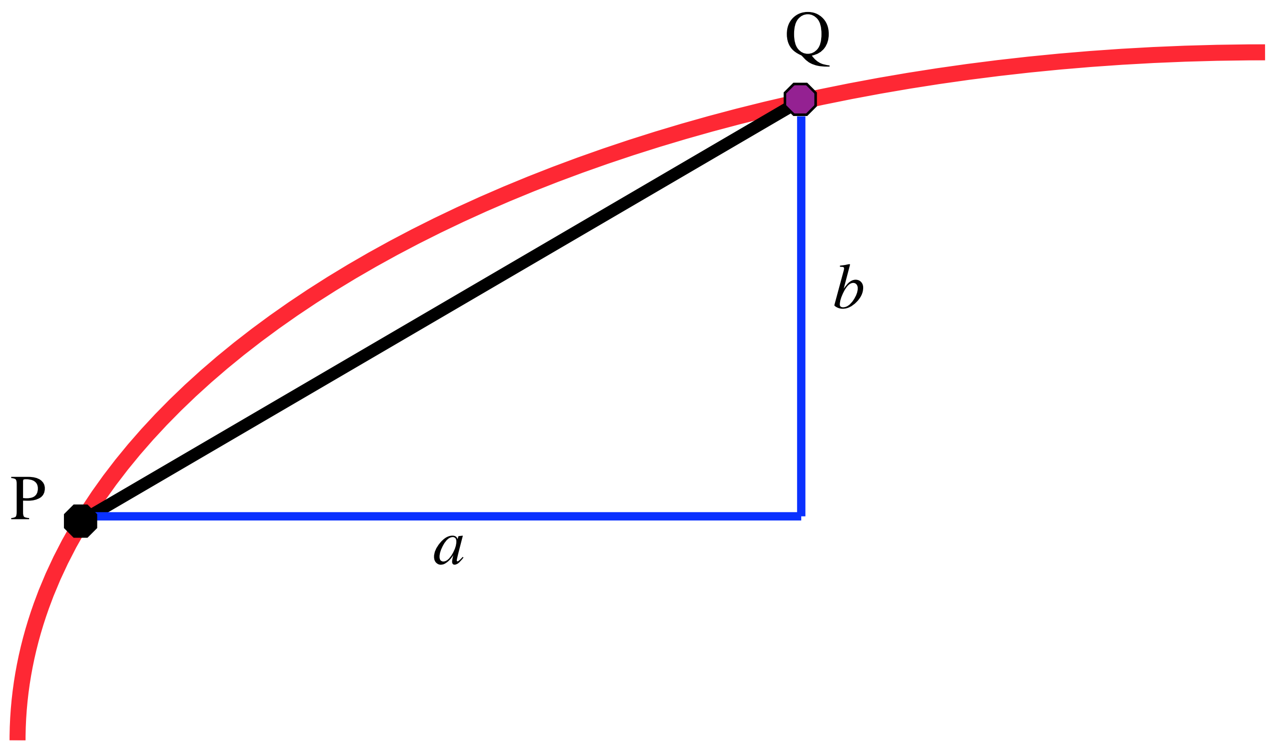
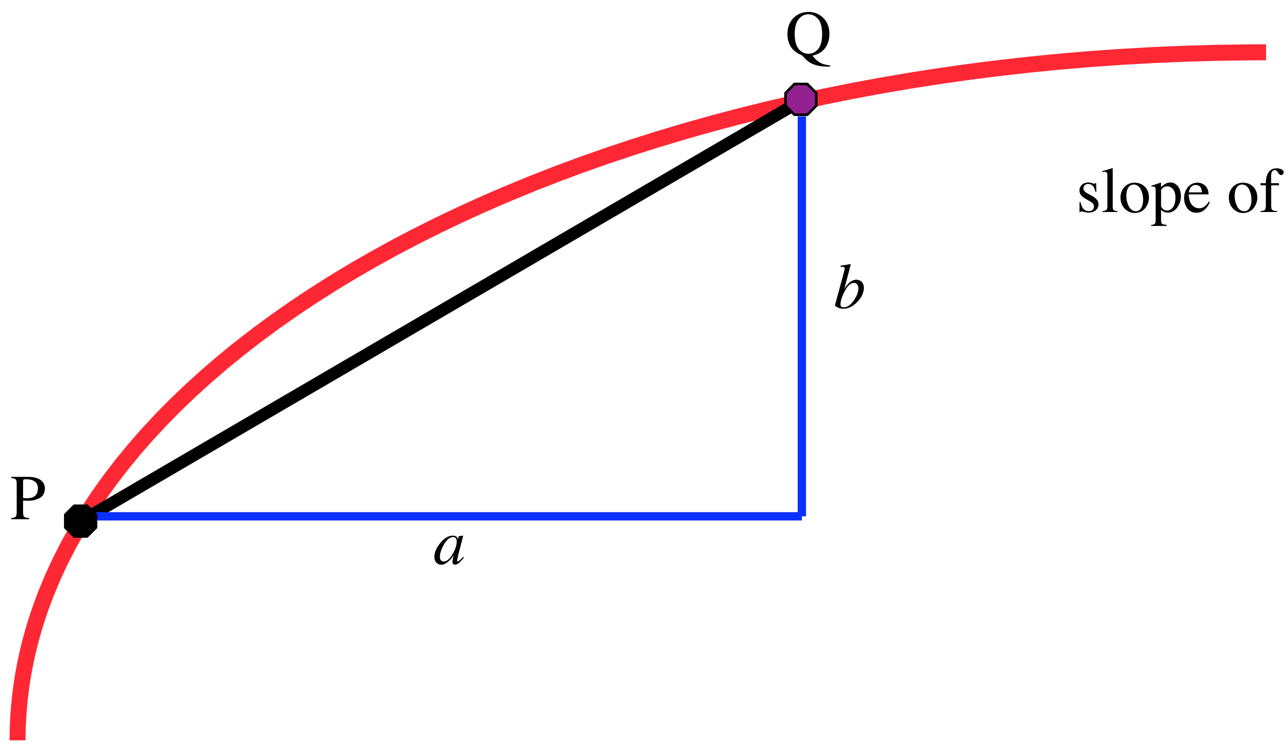
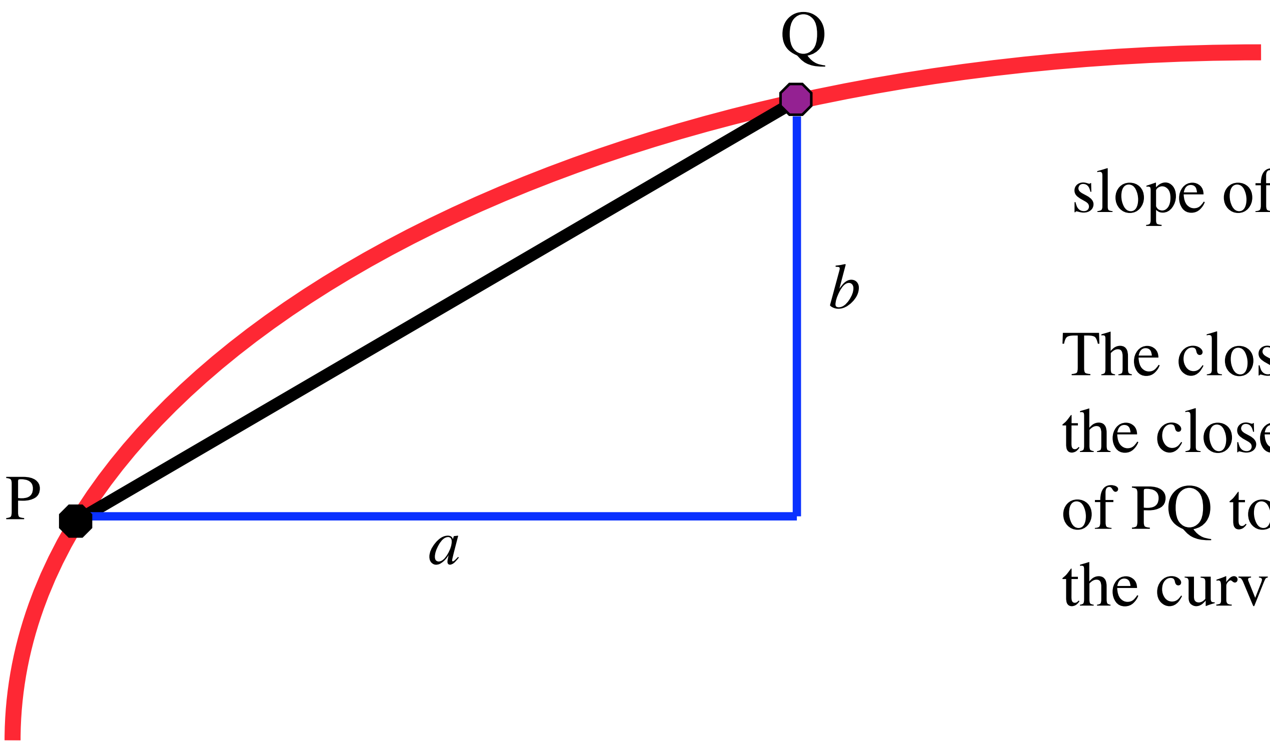


Calculating the slope at a point



slope of PQ = $\frac{b}{a}$

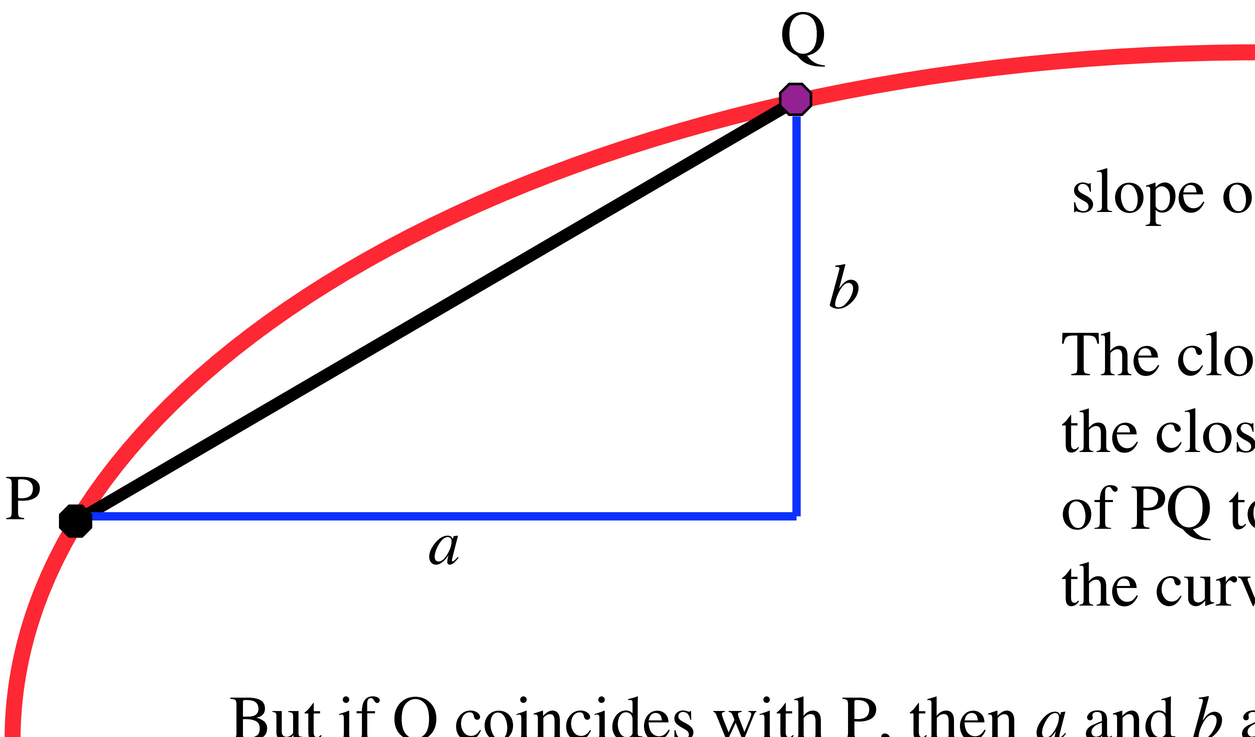
Calculating the slope at a point



slope of PQ = $\frac{b}{a}$

The closer Q is to P, the closer is the slope of PQ to the slope of the curve at P.

Calculating the slope at a point

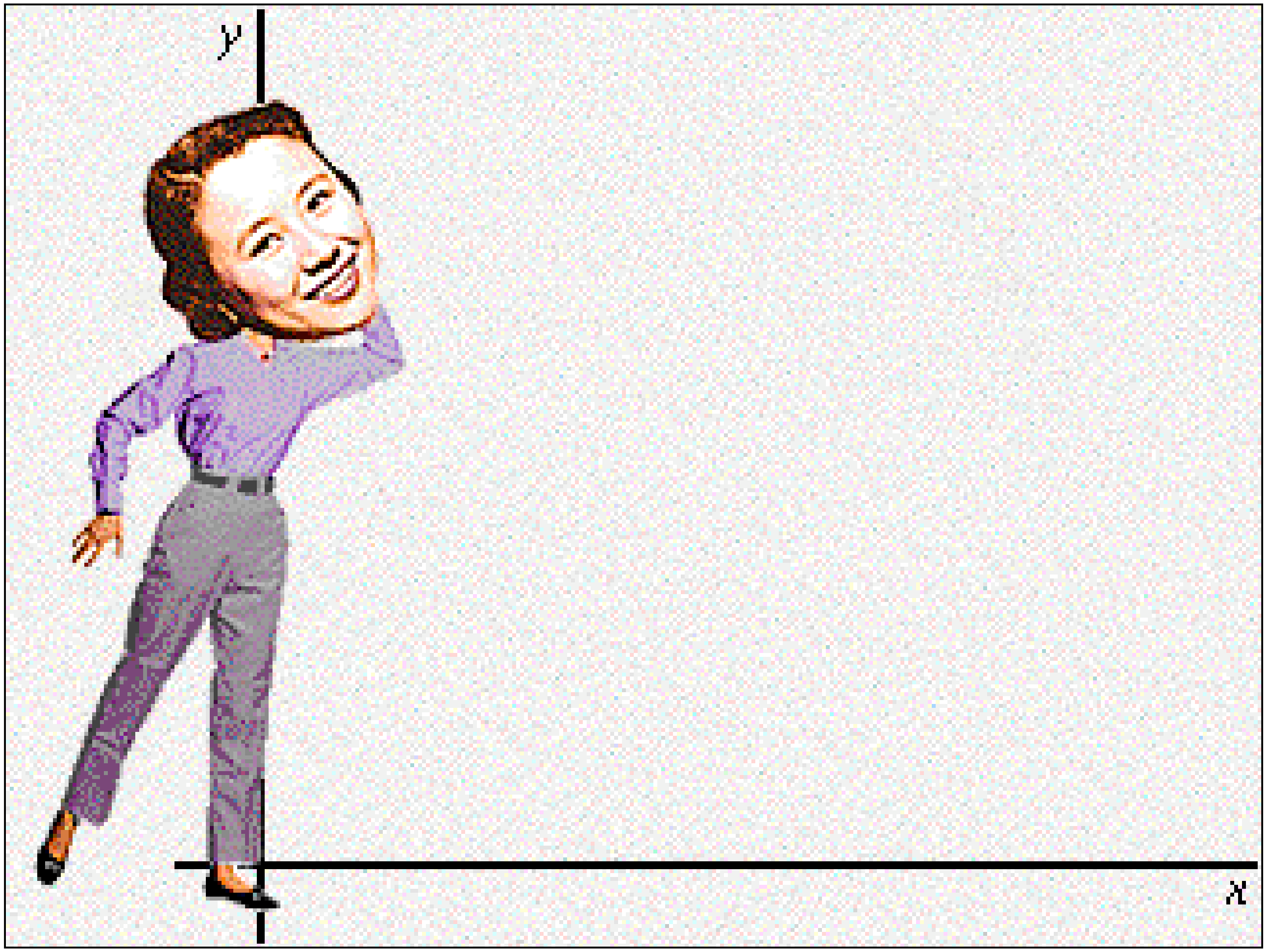


slope of PQ = $\dfrac{b}{a}$

The closer Q is to P, the closer is the slope of PQ to the slope of the curve at P.

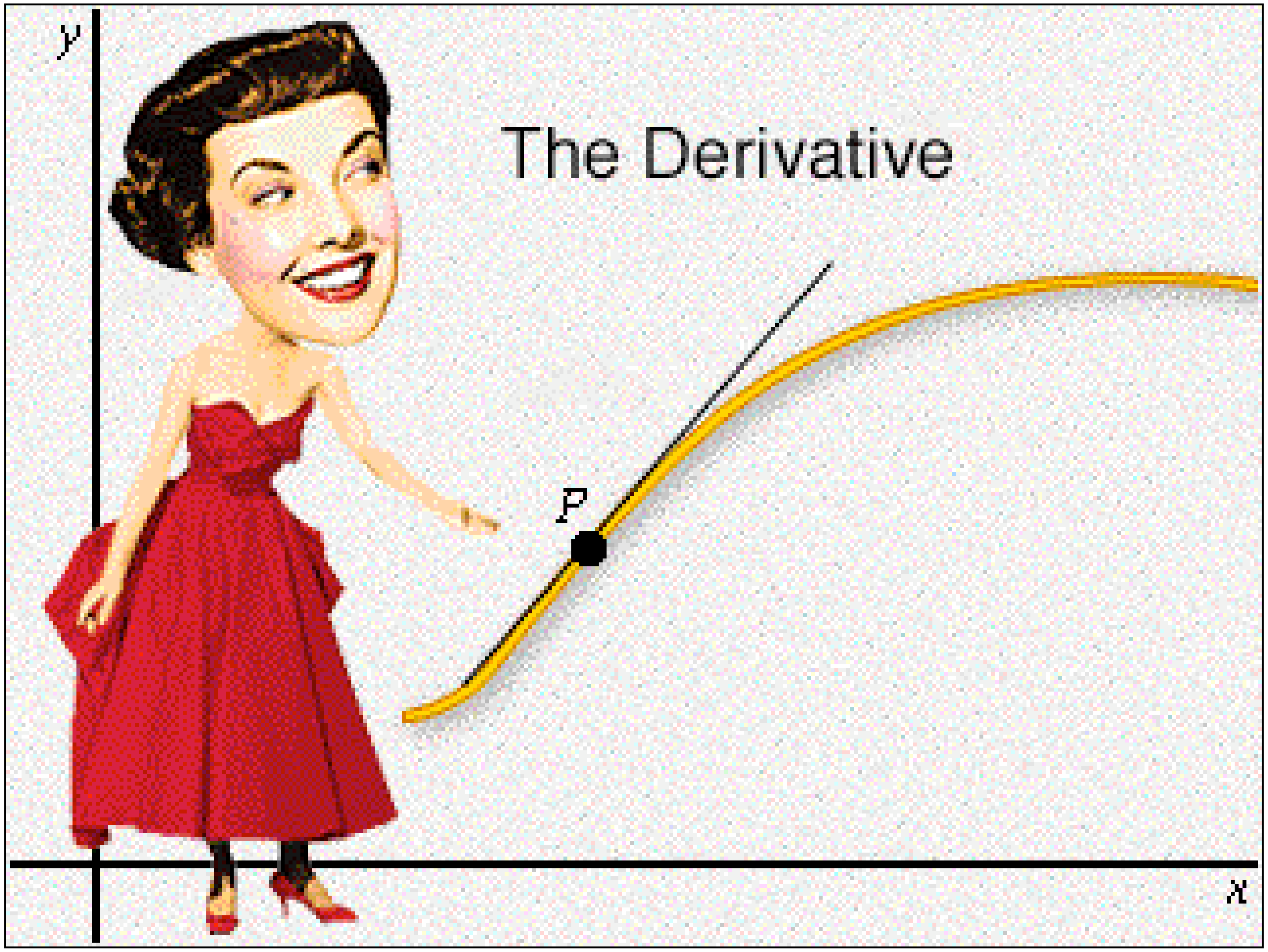
But if Q coincides with P, then *a* and *b* are both 0, and the slope of PQ works out as 0/0, which is undefined.

Differential calculus



movie: derivative_1.mov

Differential calculus



movie: derivative_2.mov

Rules of differential calculus

What makes differential calculus useful (in fact, what makes it a "calculus"), is that there are easily applied, symbolic rules for calculating the derivatives $f'(x)$ of common functions $f(x)$.

$x^n \rightarrow n\, x^{n-1}$	$\sin x \rightarrow \cos x$	$\cos x \rightarrow -\sin x$
$\tan x \rightarrow \sec^2 x$	$a^x \rightarrow (\ln a)\, a^x$	$\ln x \rightarrow 1/x$
$K\, f(x) \rightarrow K\, f'(x)$	$f + g \rightarrow f' + g'$	$fg \rightarrow fg' + gf'$

etc.

Where the rules come from

The rules are derived by making a subtle change in perspective. Having started by looking at the slope (i.e., the **pattern** of **change**) of a function f(x), you shift to looking at the **pattern** exhibited by the **slope approximations**

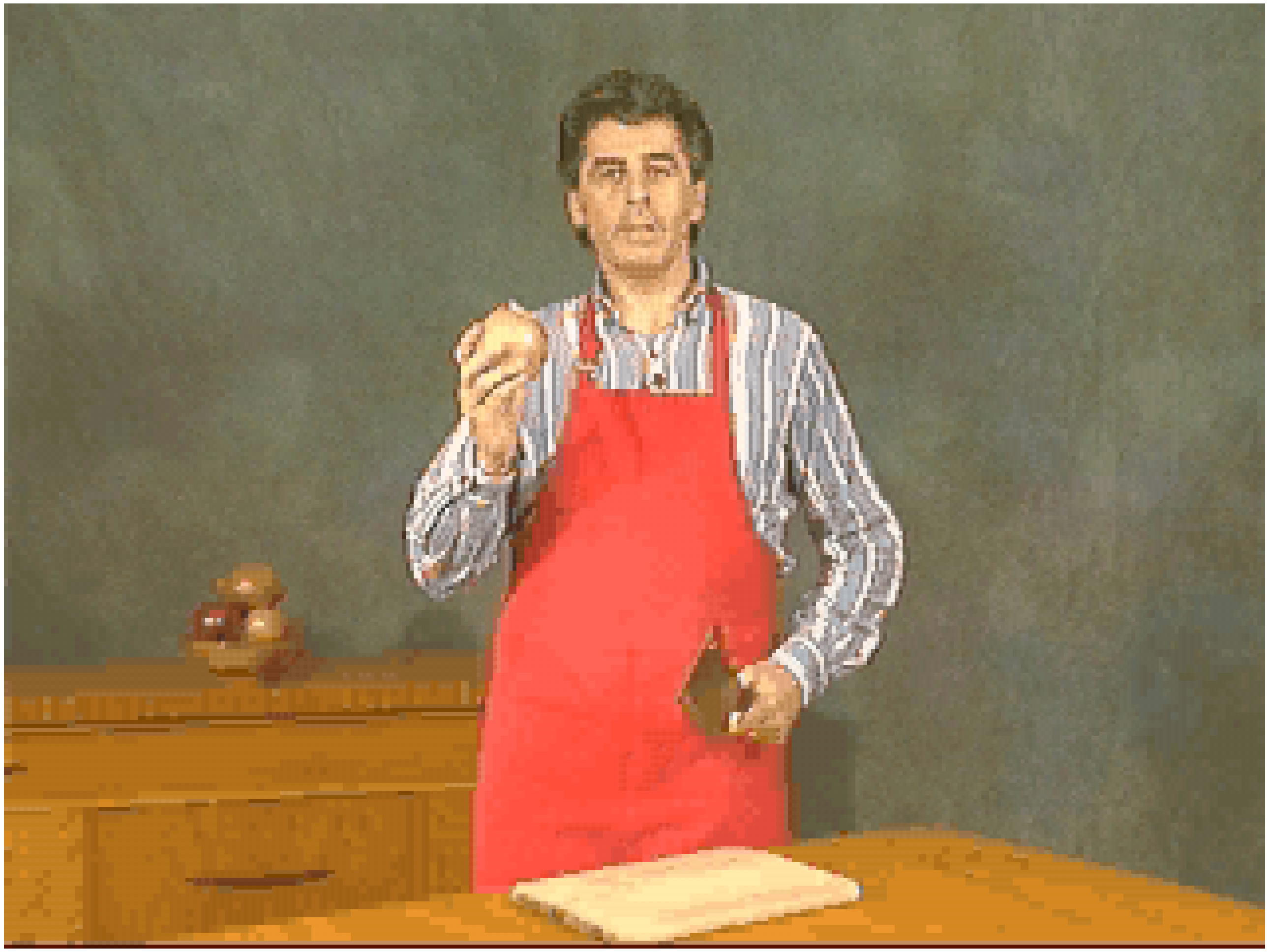
$$\frac{f(x+h) - f(x)}{h}$$

as h gets progressively smaller, and extrapolating from that pattern the limit of that sequence of approximations.

Integral calculus

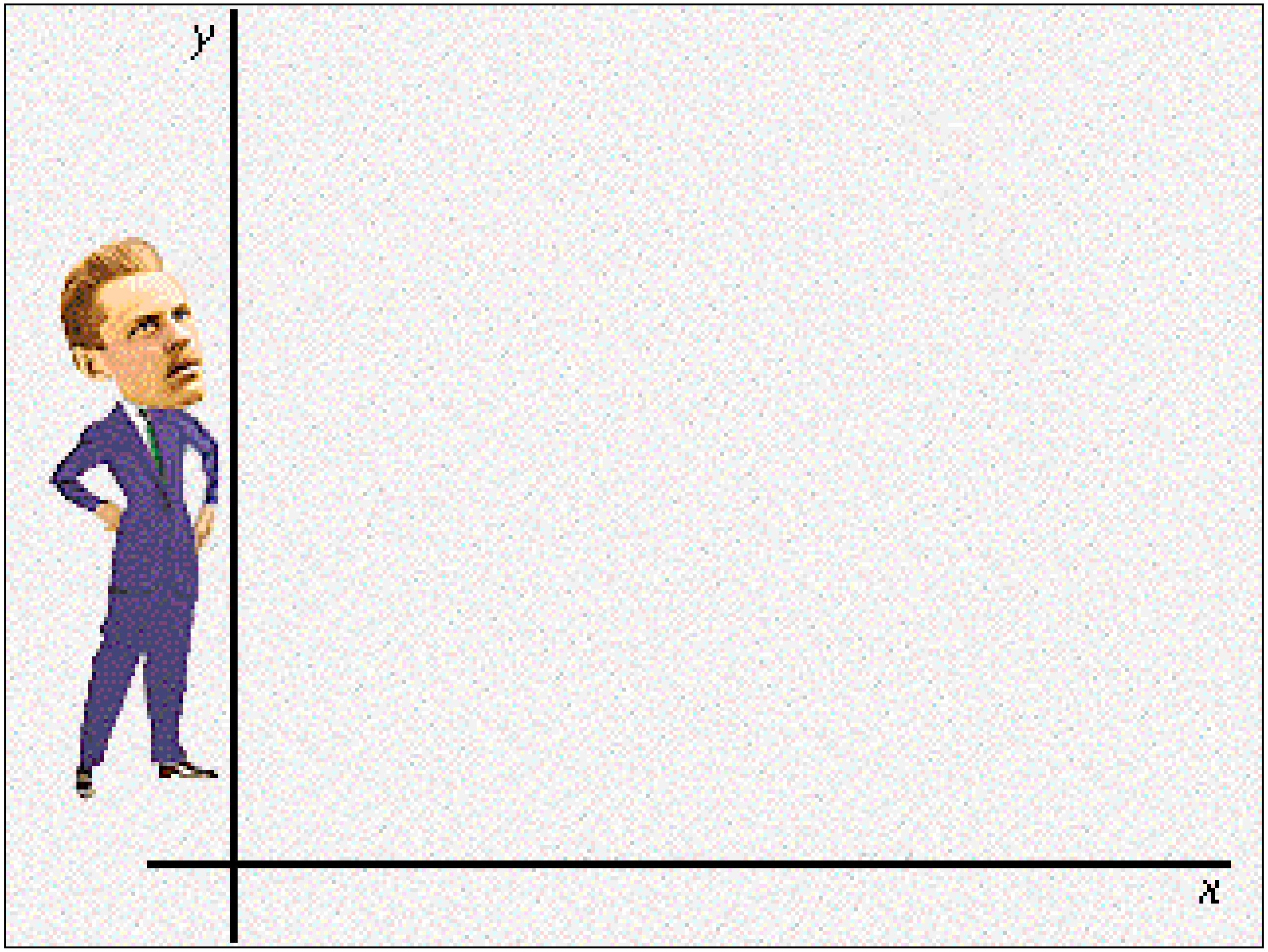
Calculating areas and volumes
of objects whose boundaries
are continuously changing

Integral calculus



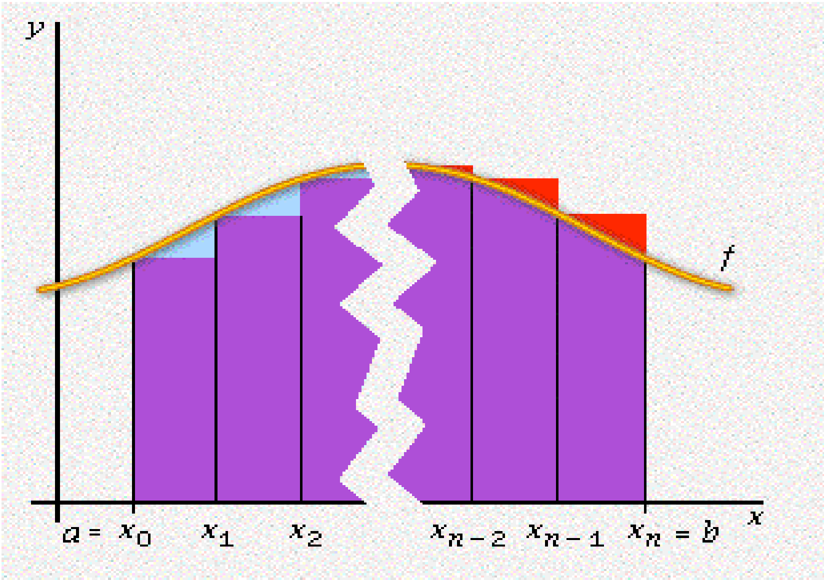
movie: integration_1.mov

Calculating the area beneath a curve



movie: integration_2.mov

The integral



$h = x_{i+1} - x_i$

Area

$= f(x_0)h + f(x_1)h + \ldots + f(x_{n-1})h$

$= [f(x_0) + f(x_1) + \ldots + f(x_{n-1})]\, h$

$= \sum_{i=0}^{n-1} f(x_i)\, h$

Calculate this value for larger and larger values of n (smaller and smaller values of h), and the limiting value gives the exact area. It is called the (definite) integral of the function f(x) from a to b:

$$\int_a^b f(x)\, dx$$

The Fundamental Theorem of Calculus

If the derivative of $f(x)$ is $g(x)$,
then the integral of $g(x)$ is $f(x)$.

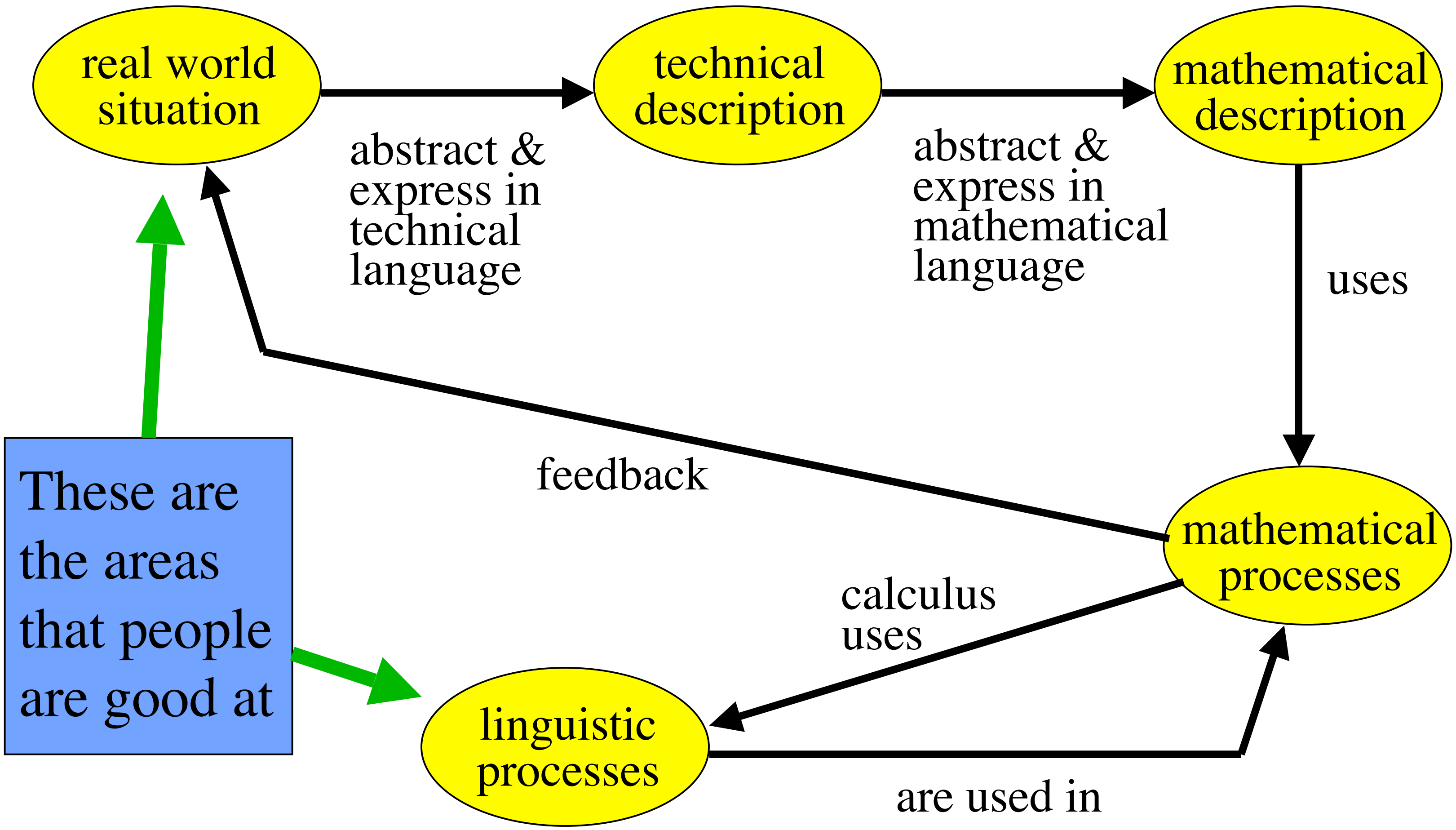
Hence, for integration we have the symbolic rules:

$x^n \rightarrow x^{n+1}/(n+1)$ $\quad$ $\cos x \rightarrow \sin x$ $\quad$ $\sin x \rightarrow -\cos x$

$a^x \rightarrow a^x/(\ln a)$ $\quad$ $1/x \rightarrow \ln x$ $\quad$ etc.

together with various rules for integrating combinations of functions.

Why calculus is a successful technology



The patterns that calculus builds on

- In the world: something moving continuously (e.g. a planet)
- Abstracted to a graph, captured by a function f.
- Mathematical model: Slope (a pattern of graphs) is associated to velocity (a pattern of moving objects)
- Change in slope (another pattern of graphs, also a pattern of slopes of graphs) is associated with acceleration (another pattern of moving objects)
- The derivative function f′ captures the way the slope changes with the position.
- To compute f′ you have to shift attention from the behavioral pattern of f to the behavioral pattern of the quantity

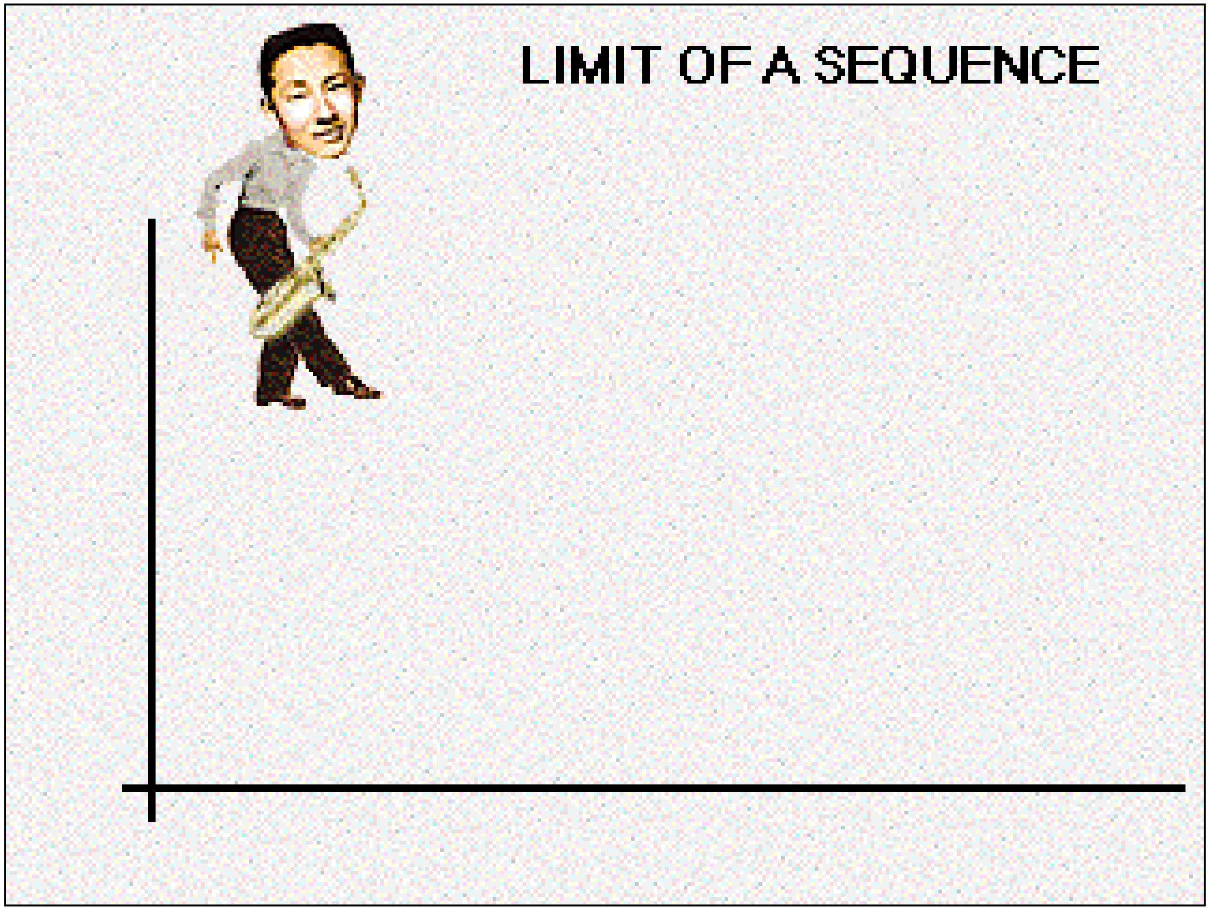
  $$\frac{f(x+h) - f(x)}{h}$$

  as h approaches 0. (The approximations to the slope at x.)
- This entails examining such "sequences of approximations behavior" patterns.

Patterns of endless number sequences

What does it mean to say that a sequence s_n of numbers approaches a limit L as n increases indefinitely?

Patterns of endless number sequences



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Limit of a number sequence

Definition: The number sequence $\{s_n\}$ has limit L as $n \to \infty$ if, for any given positive real number ε, there is a number N such that $|s_n - L| < \varepsilon$, whenever $n \geq N$.

This replaces a dynamic concept with a static definition.

The patterns of calculus

Notice that as we develop calculus, at each stage we replace a dynamic pattern with a static one:

- Continuous motion is replaced by a (static) graph.
- The process of approximating the slope is replaced by the determination of the limit of a sequence of numbers.
- The dynamic aspect of moving along a number sequence is replaced by the question of whether certain numbers exist having particular properties.

The patterns of calculus

Other concepts that had to be developed were:

- What it means to say that a function f(x) has **limit** L as $x \to a$ (for some fixed number a).
- What it means to say that a function f(x) is **continuous**.

Continuity

- First defined for functions from real numbers to real numbers.
- Then defined for metric spaces.
- Then defined for topological spaces.

Metric spaces

Any set M together with a function

d: M x M → R, having the properties

(i) d(x,y) ≥ 0 for all x,y in M

(ii) d(x,y) = 0 if and only if x = y

(iii) d(x,y) = d(y,x)

(iv) d(x,z) ≤ d(x,y) + d(y,z)

Topological spaces

Any set T together with a collection $\mathcal{C}$ of subsets of T (called the **open** sets of the topology), having the properties:

(i) $\varnothing$ and T are in $\mathcal{C}$

(ii) if X, Y are in $\mathcal{C}$, then $X \cap Y$ is in $\mathcal{C}$

(iii) if $\mathcal{F}$ is any collection of members of $\mathcal{C}$, then $\cup\mathcal{F}$ is in $\mathcal{C}$.

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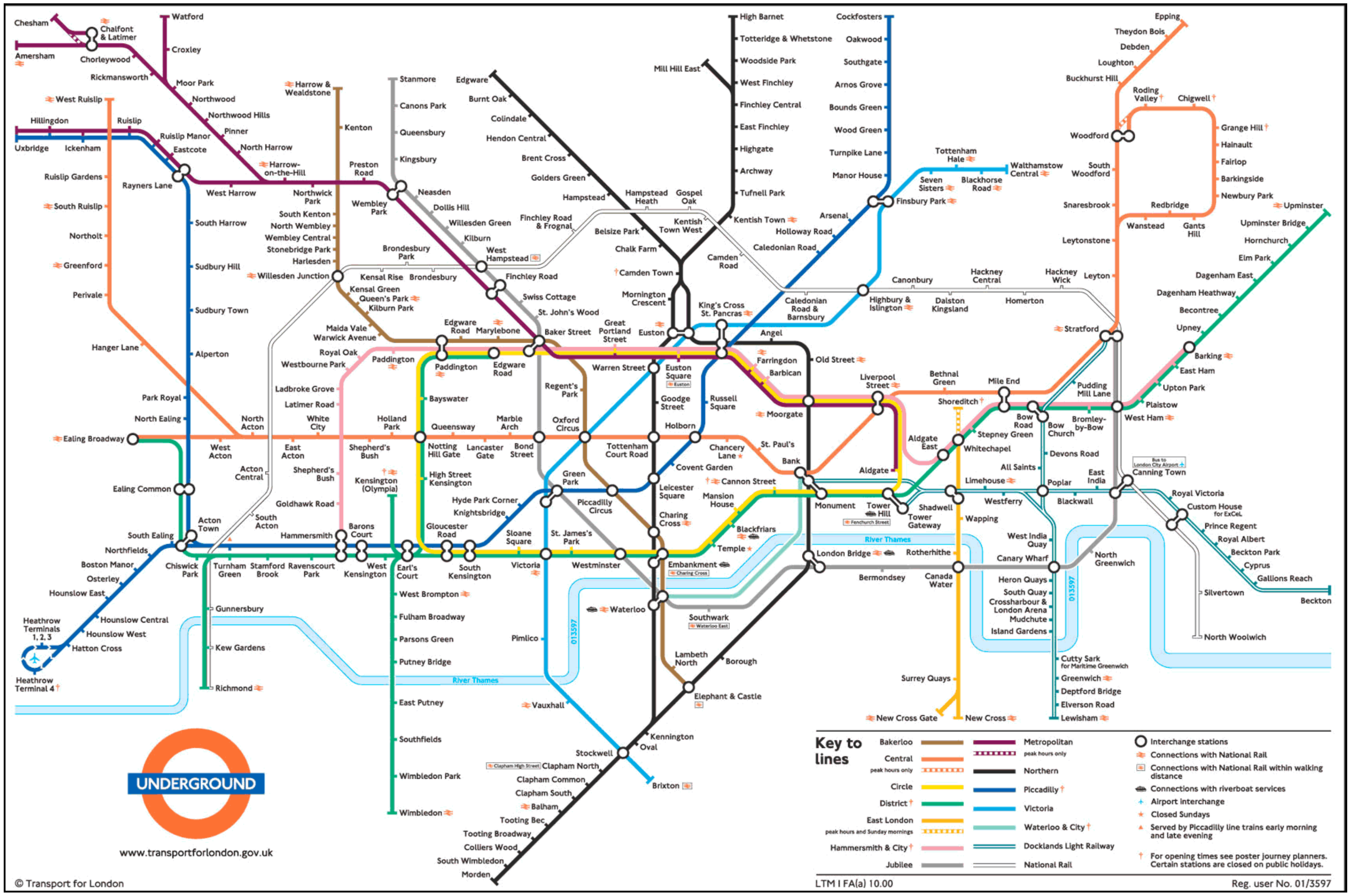
A function f from one topological space X to another one Y is **continuous** if $f^{-1}[U]$ is an open set in X, for every open set U in Y.

Topology of surfaces

(Rubber sheet geometry)



Euler’s solution to the Königsberg Bridges Problem

A classical result

in topology

Map of Kaliningrad

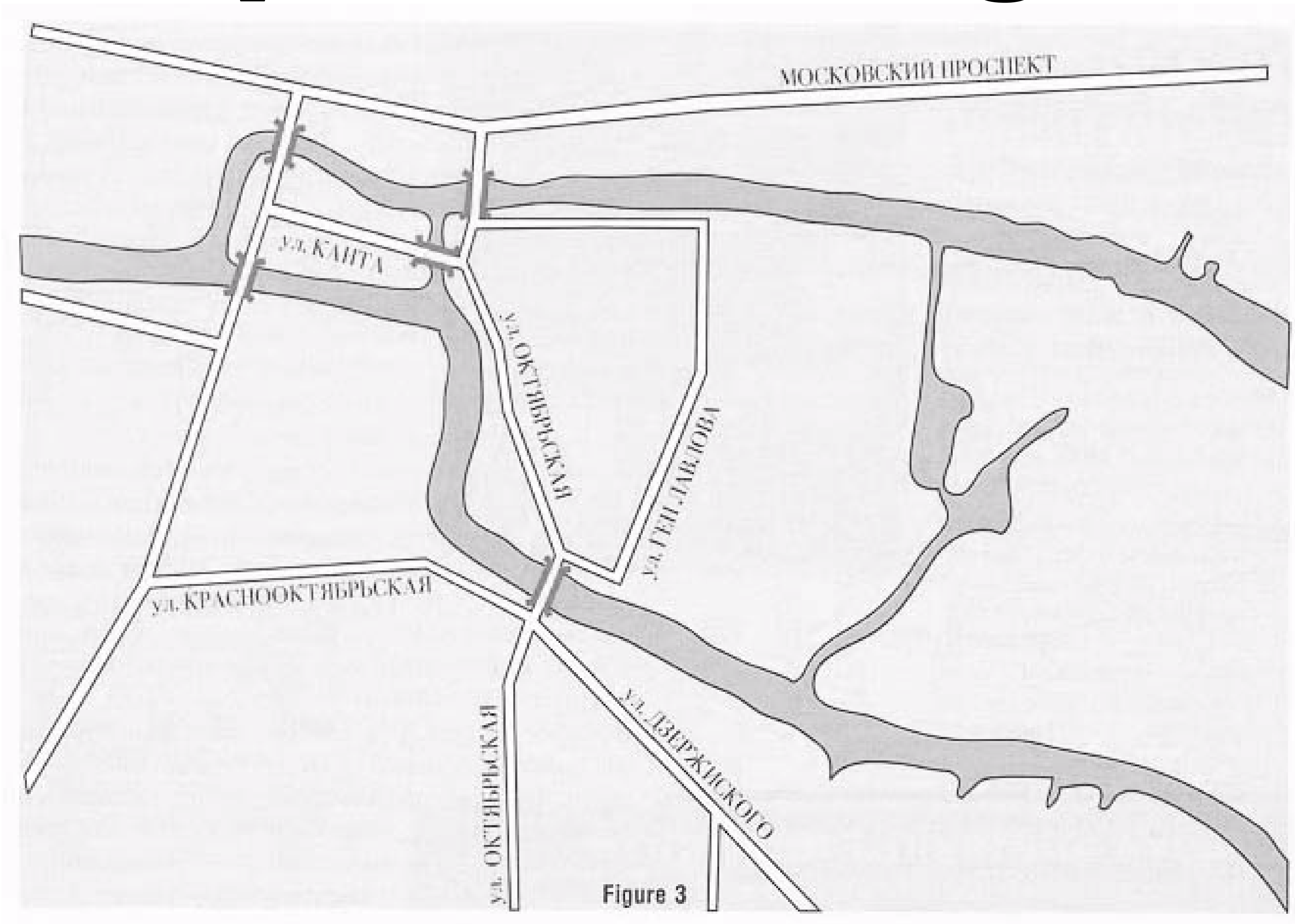


Figure 3

Map of Königsberg

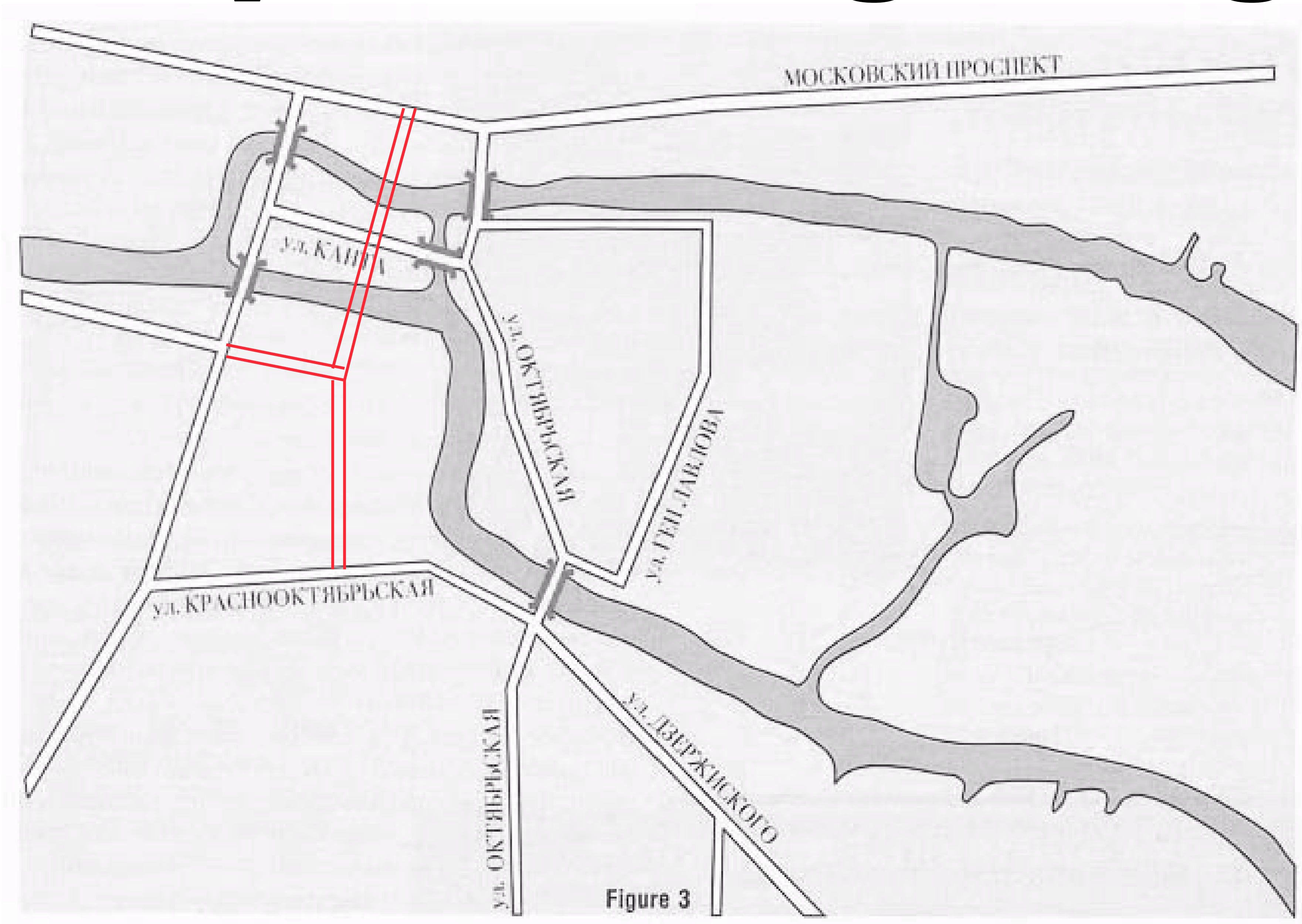


Figure 3

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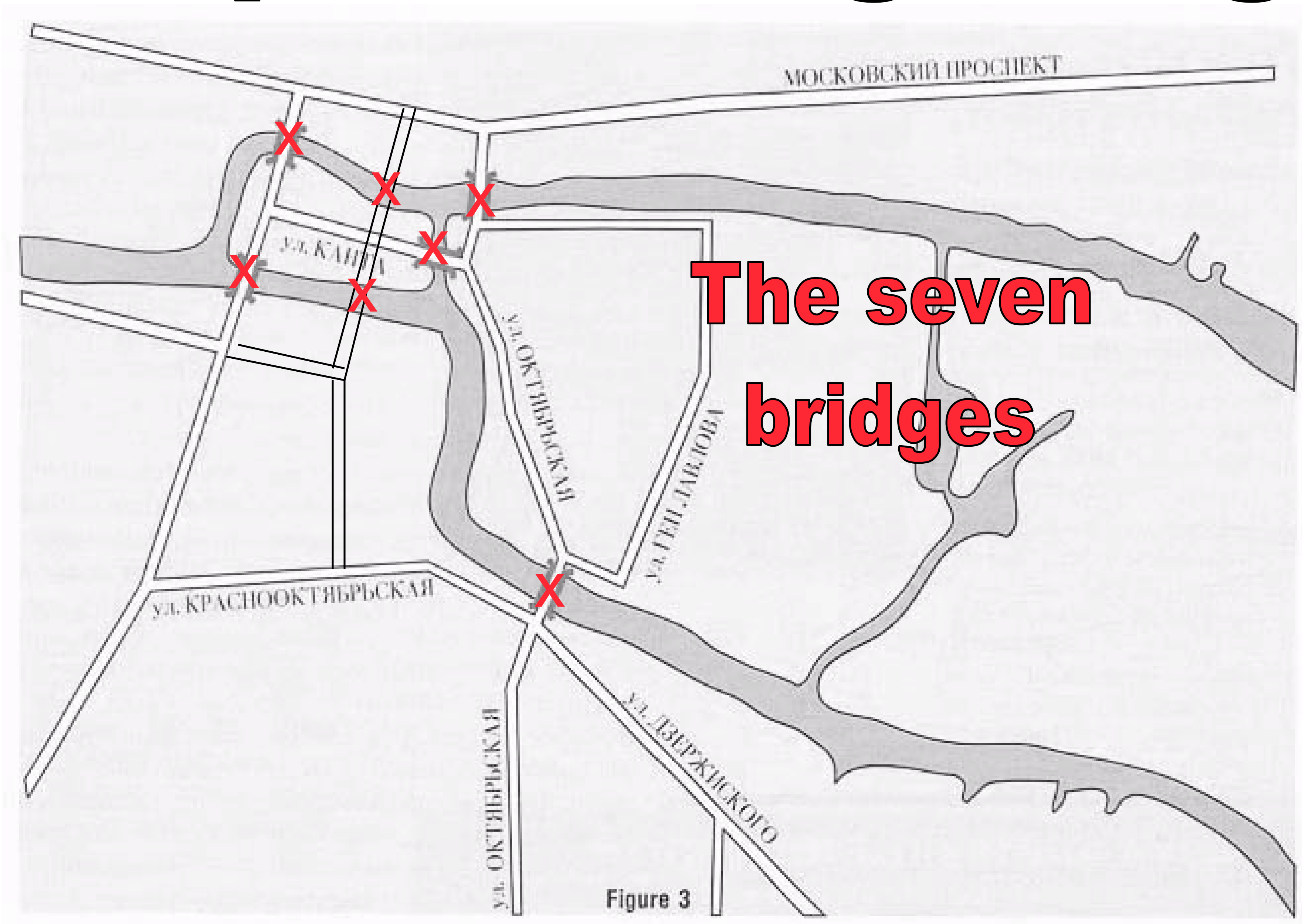
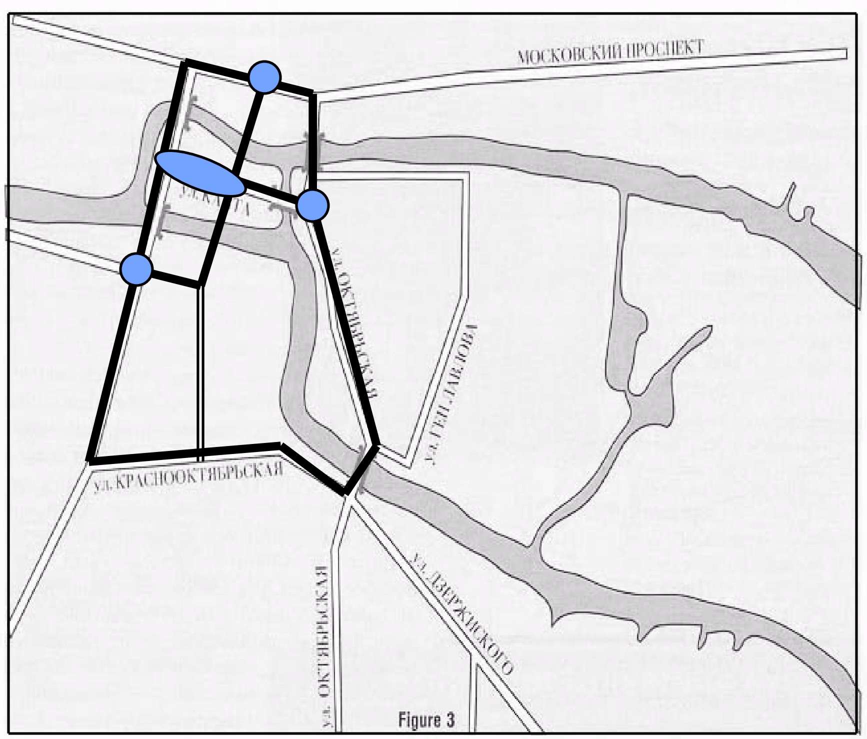


Euler's network representation of the Königsberg bridges



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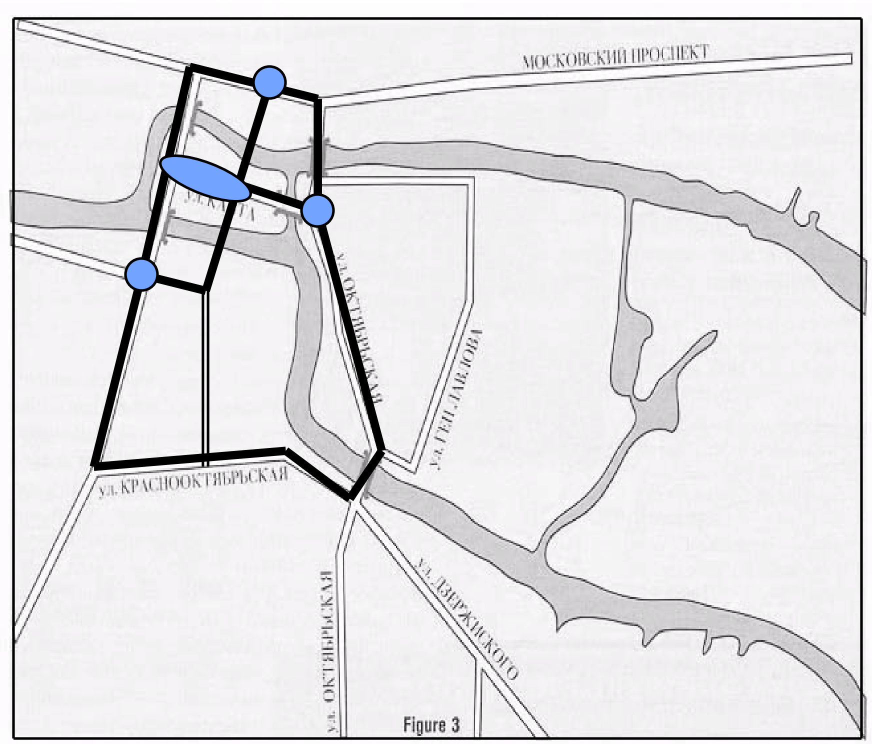
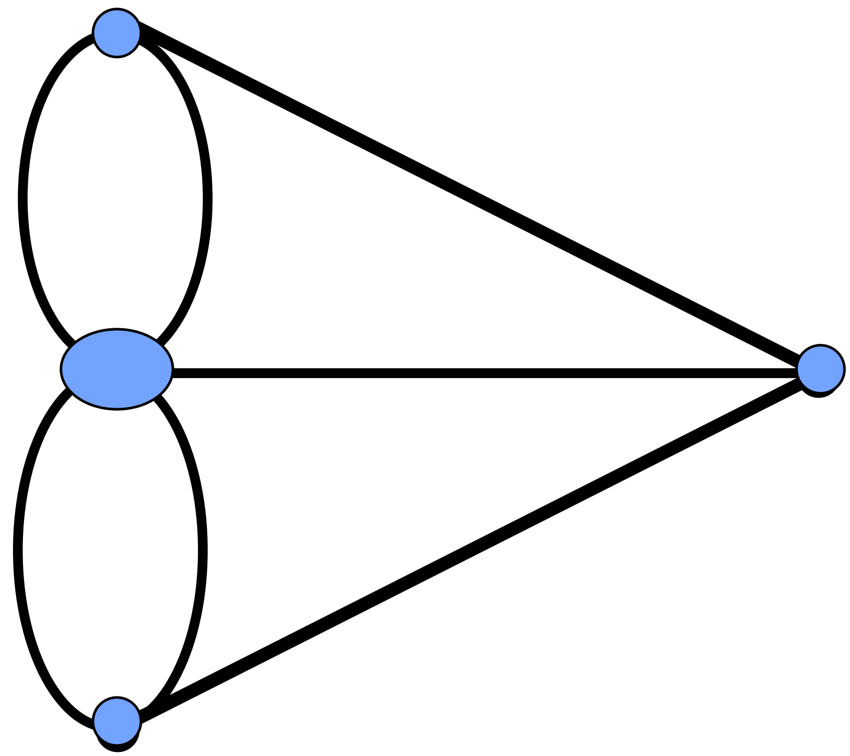




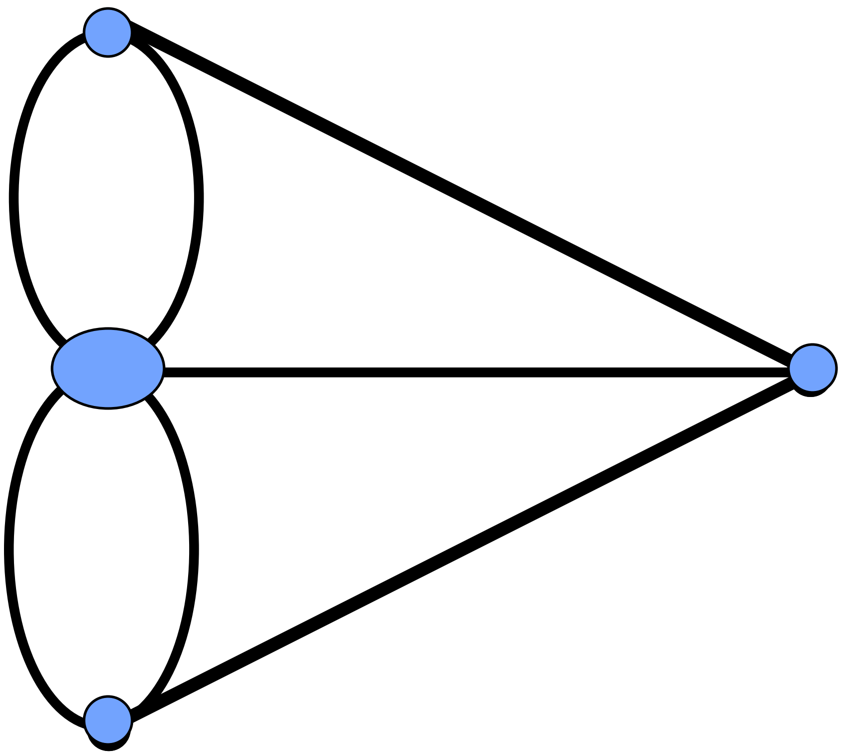
Euler's network theorem

For any network drawn in the plane, if V denotes the number of vertices, E the number of edges, and F the number of faces (enclosed regions), then

$$V - E + F = 1$$

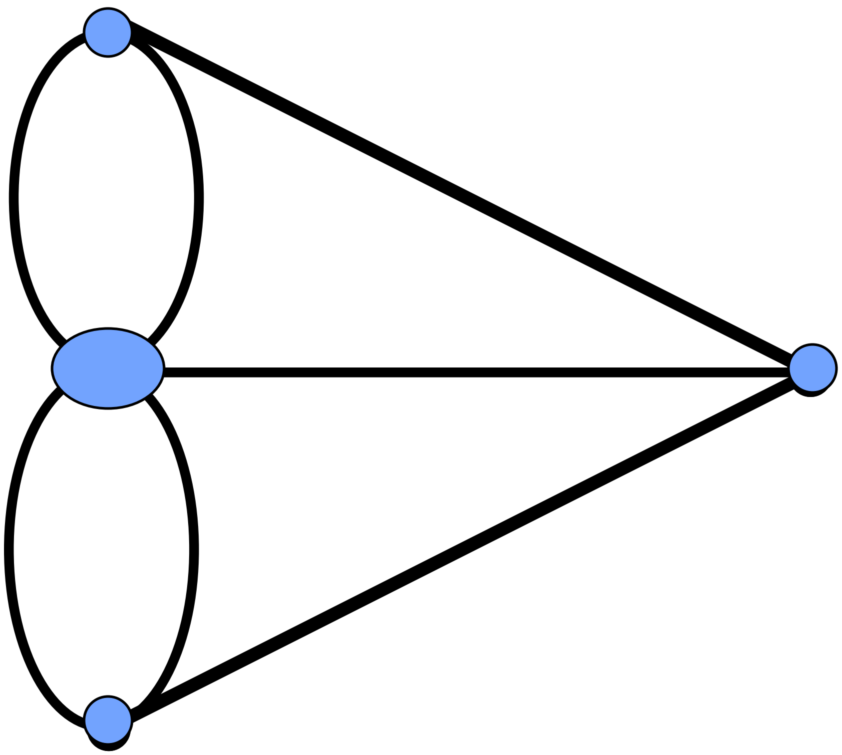
Euler's network theorem

Example



Euler’s network theorem

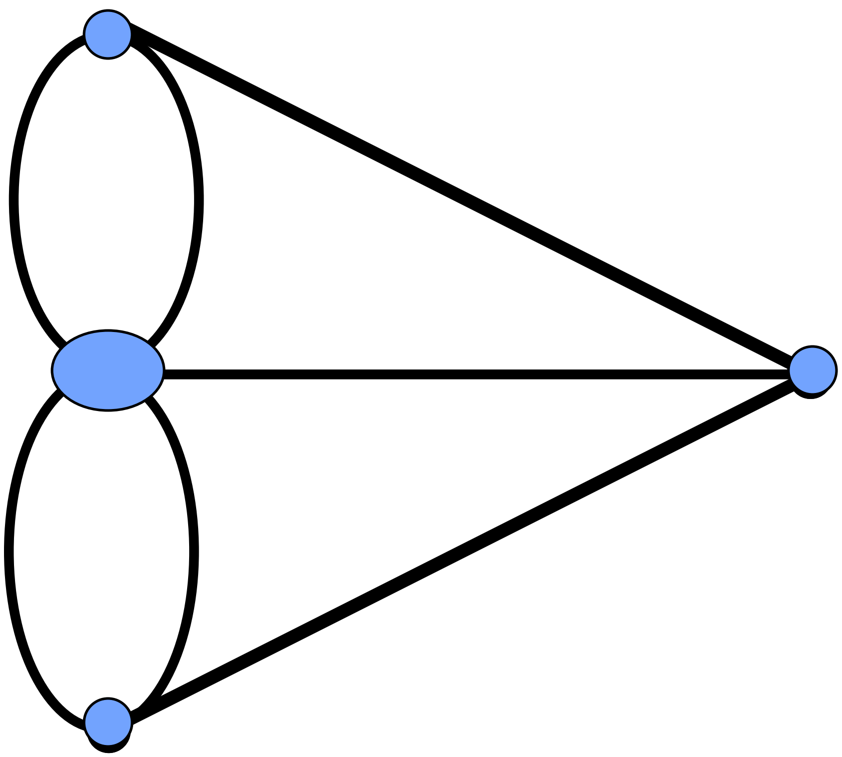
Example



V = 4

Euler's network theorem

Example

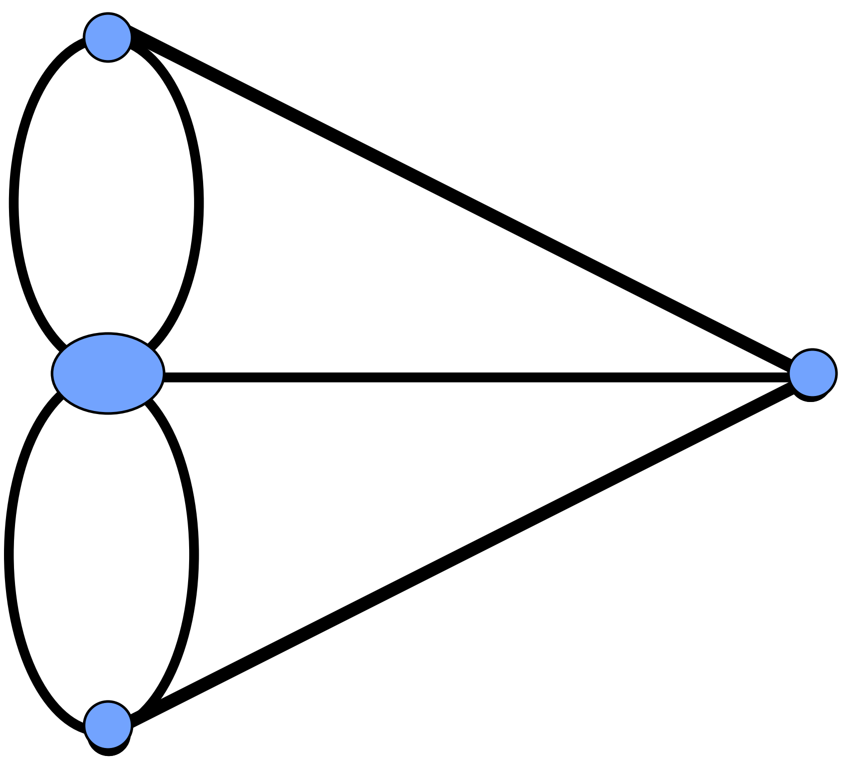


V = 4

E = 7

Euler's network theorem

Example

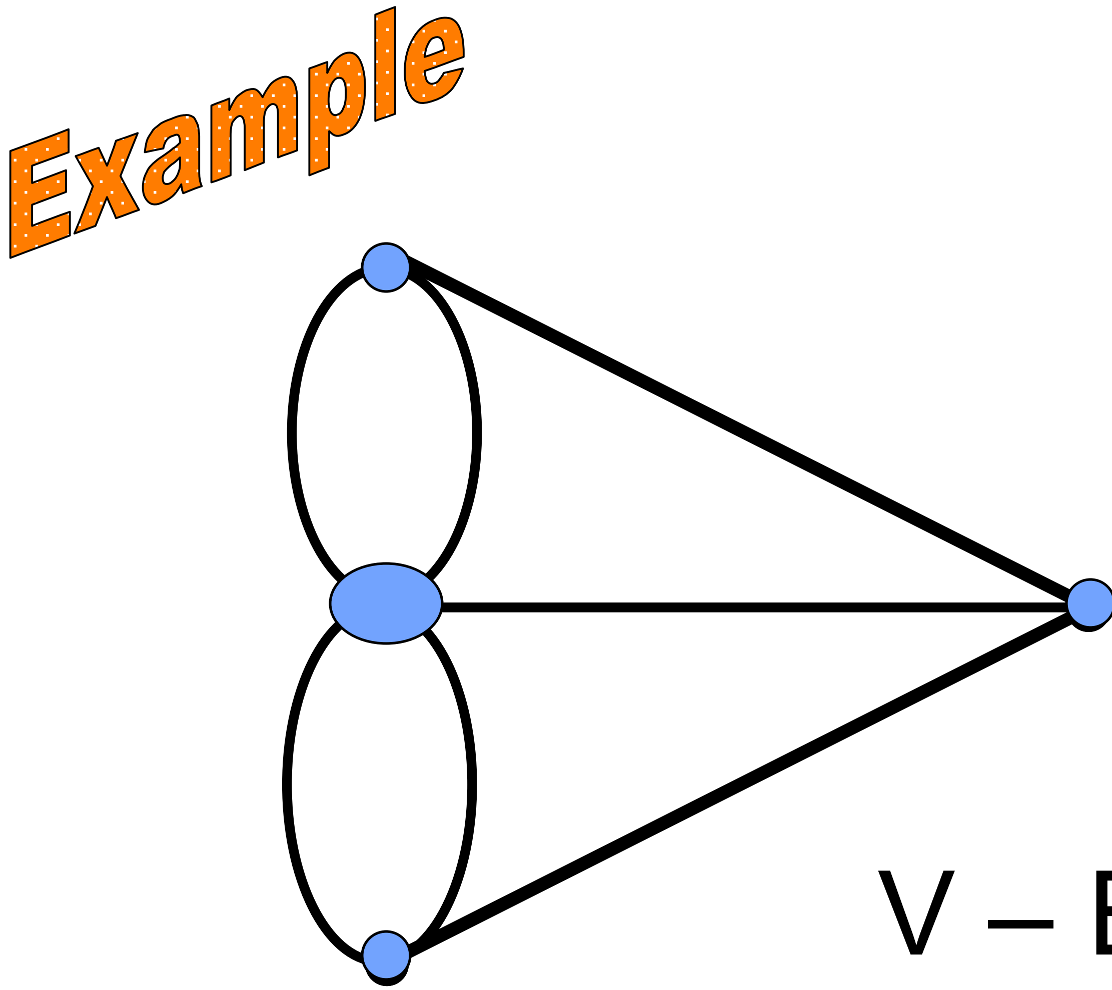


V = 4

E = 7

F = 4

Euler's network theorem



V = 4

E = 7

F = 4

V – E + F = 1