Stanford Continuing Studies, Course MATH 07, Fall 2005

Three views of mathematics

LECTURE 3

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(Pure) mathematics through the eyes of the (pure) mathematician

- A body of study having aesthetic appeal
- An intellectual challenge
- Abstract, context free
- Axiom based
- Results established by rigorous proofs
- Typical problem: “Prove that … ”

Famous challenge problems of the twentieth century

- Fermat's last theorem: Prove that for any integer exponent n greater than 2, the equation $x^n + y^n = z^n$ has no nonzero integer solutions
- The four color theorem: Prove that any map drawn in the plane may be colored using at most four colors
- Classification of finite simple groups: Describe the different kinds of finite simple groups

The Millennium Problems

The Prize:

$7,000,000

The math:

$$7 \times \$1{,}000{,}000 = \$7{,}000{,}000$$

The announcement

24 May, 2000, Collège de France, Paris

Sir Michael Atiyah of Great Britain and John Tate of the USA:

"A prize of $1 million will be awarded to the person or persons who first solves any one of seven of the most difficult open problems of mathematics.

These problems will henceforth be known as the Millennium Problems."

CLAY MATHEMATICS INSTITUTE

dedicated to increasing and disseminating mathematical knowledge

A Celebration of the Universality of Mathematical Thought

Michael Atiyah
Timothy Gowers
John Tate

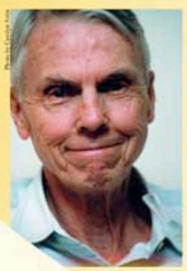
Wednesday, May 24th, 2000

2:00 pm

Collège de France

11, place Marcelin Bertelot, 75005 Paris

Amphithéâtre Marguerite de Navarre



2:00 pm to 6:00 pm
  Clay Mathematics Award
  Keynote Address: Timothy Gowers
  Millennium Prize Problems: John Tate, Michael Atiyah
6:00 pm to 7:00 pm: Reception

Continuation on May 25th, at 9:30 am, with talks by
M. Bhargava, D. Gaitsgory, L. Lafforgue, and T. Tao

OPEN TO THE PUBLIC Further information: www.claymath.org
Phone: 1-617-868-8277 or +33 (0)1 44 27 17 08

The inspiration ...

The Hilbert Problems

A list of 23 unsolved mathematics problems announced by David Hilbert at an international mathematics meeting in 1900.

All but one have been solved in one way or another.

The Clay Mathematics Institute

Established in 1999 by Landon Clay

Based in Cambridge, MA

Initial endowment: $90 million

Founding Director: Arthur Jaffe (Harvard)

Millennium Prize Committee:
Arthur Jaffe
Andrew Wiles (Princeton)
Michael Atiyah (Cambridge, UK),
Alain Connes (Paris),
Edward Witten (Princeton)

Alain Connes, Edward Witten, Andrew Wiles, Arthur Jaffe
Landon Clay, Lavinia Clay, Finn Caspersen

May 10, 1999

Why offer the prize?

"Curiosity is part of human nature. Unfortunately, the established religions no longer provide the answers that are satisfactory, and that translates into a need for certainty and truth. And that is what makes mathematics work, makes people commit their lives to it. It is the desire for truth and the response to the beauty and elegance of mathematics that drives mathematicians."

– Landon Clay

The Millennium Problems

P versus NP

The Hodge Conjecture

The Poincaré Conjecture

The Riemann Hypothesis

Yang-Mills Existence and Mass Gap

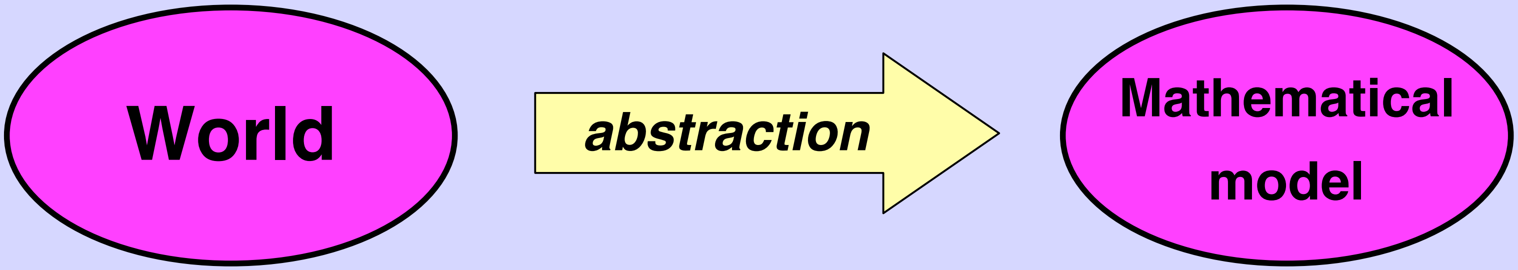
Navier-Stokes Existence and Smoothness

The Birch and Swinnerton–Dyer Conjecture

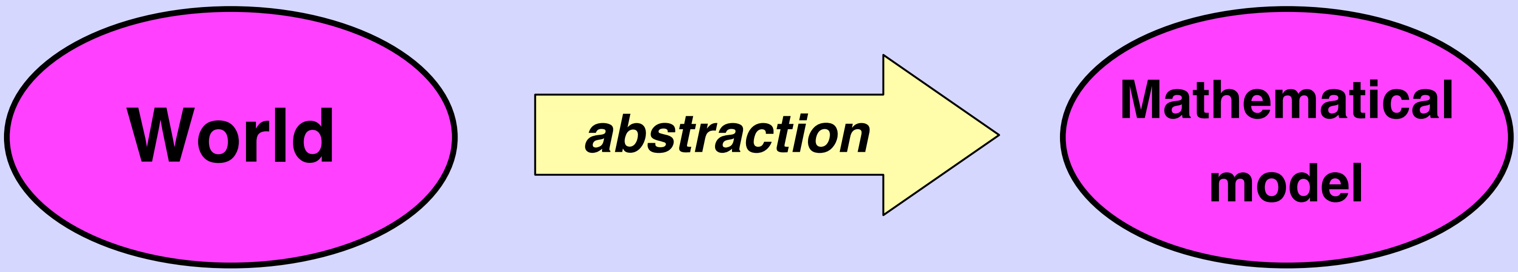
The Millennium Problems

Why are the problems so hard to describe to the layperson?

Mathematics and the world



Mathematics and the world



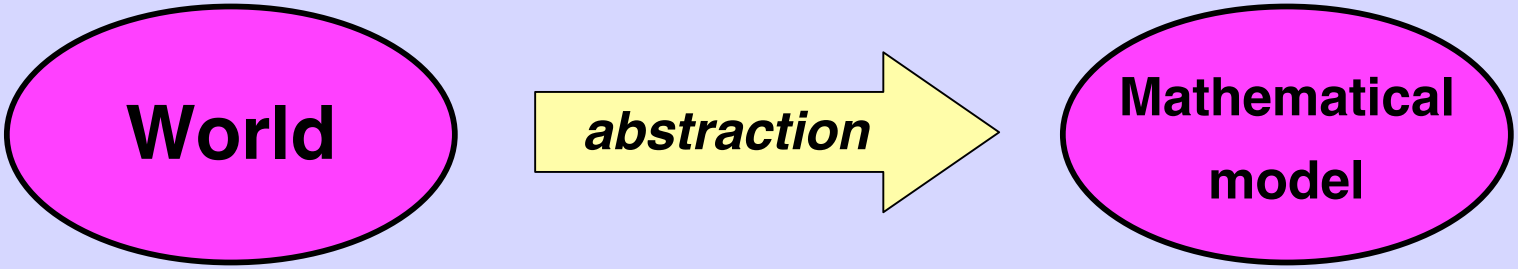
buildings
tables
chairs
airplanes
bridges
etc.

Mathematics and the world

World → *abstraction* → **Mathematical model**

buildings	**lines**
tables	**triangles**
chairs	**curves**
airplanes	**numbers**
bridges	**equations**
etc.	**etc.**

Mathematics and the world



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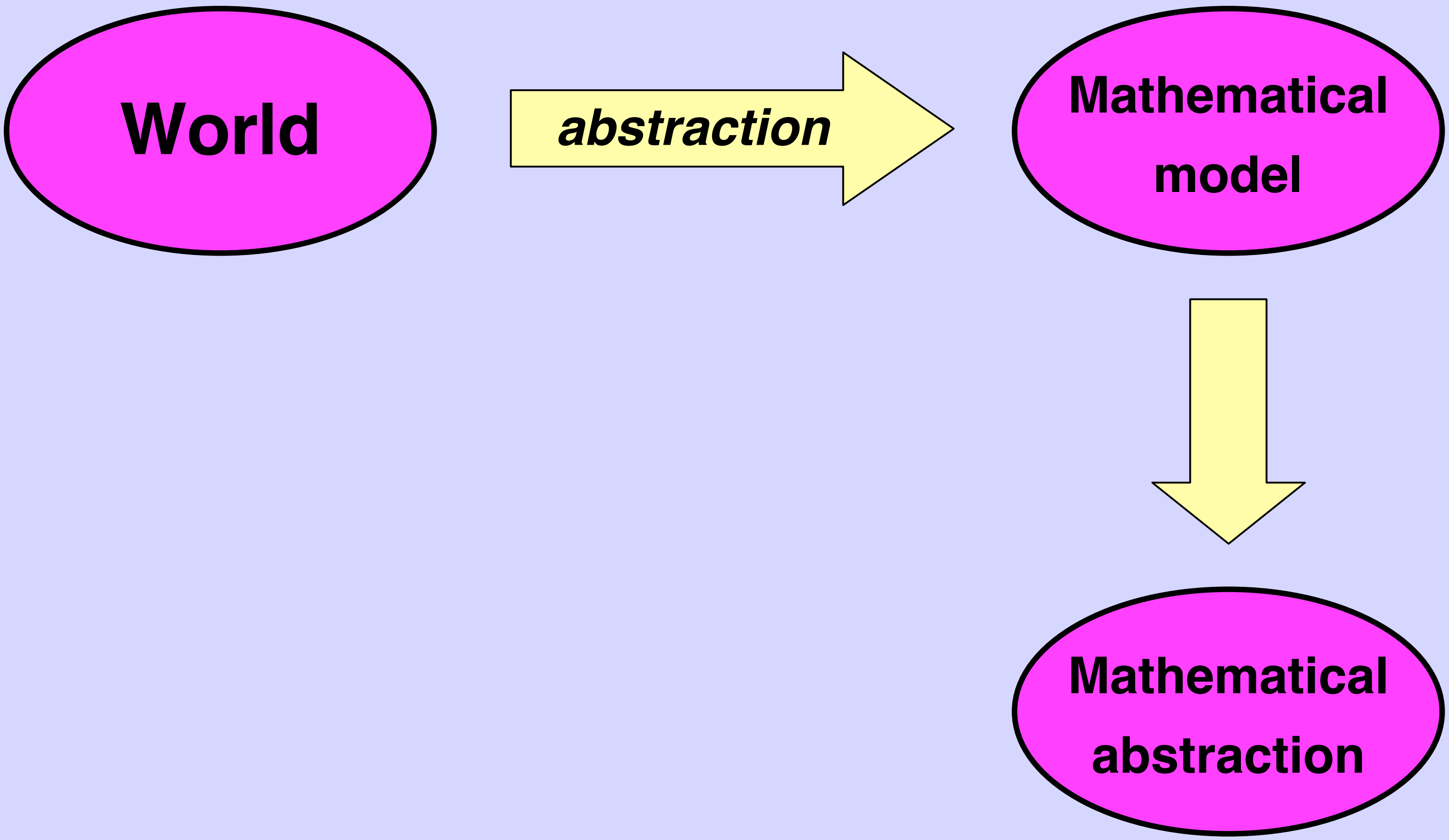

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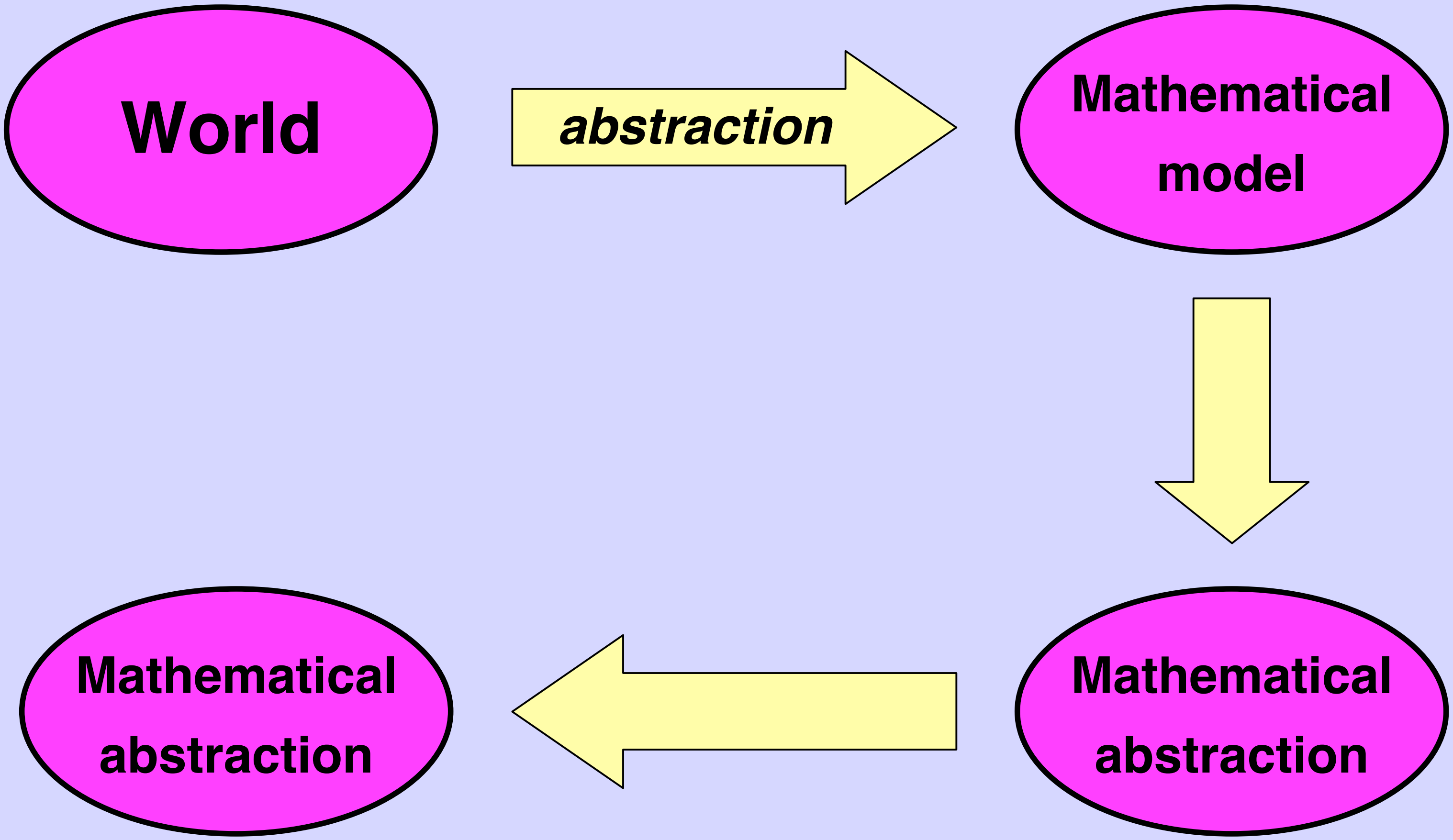

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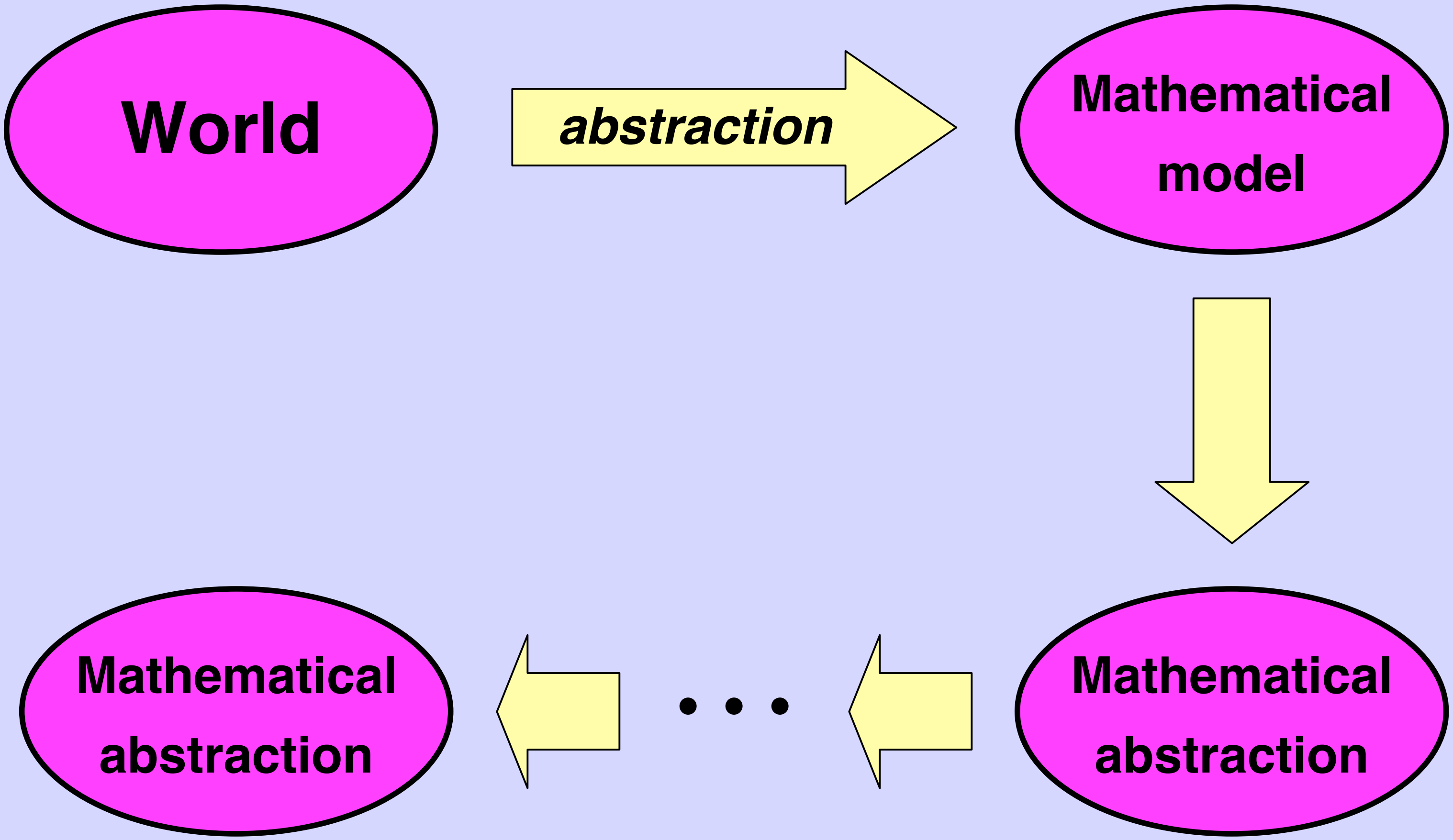

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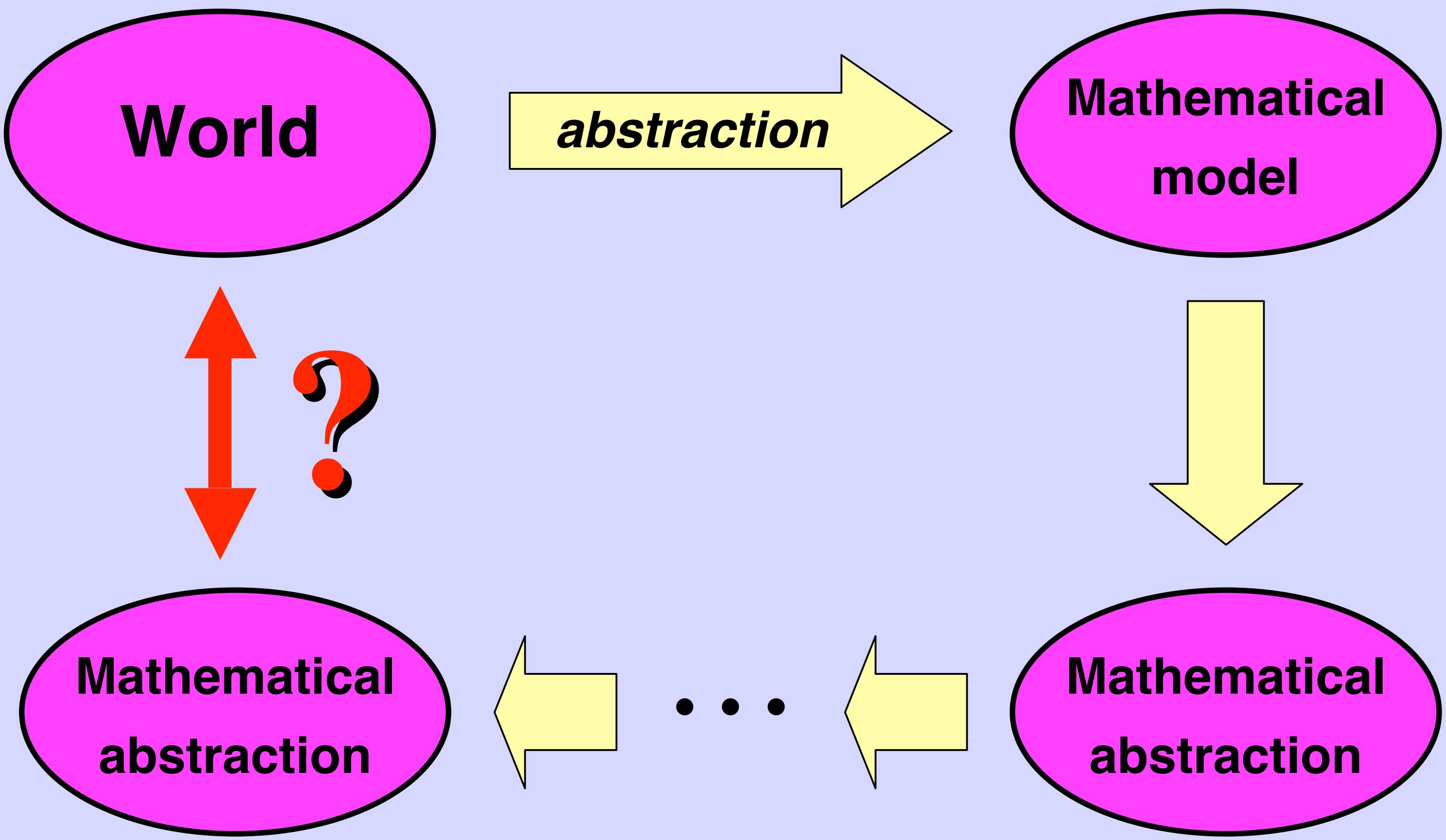


The Millennium Problems

P versus NP

The Hodge Conjecture

The Poincaré Conjecture

The Riemann Hypothesis

Yang-Mills Existence and Mass Gap

Navier-Stokes Existence and Smoothness

The Birch and Swinnerton–Dyer Conjecture

The Riemann Problem

- Formulated in 1859
  by Bernhard Riemann.
- Was on Hilbert’s list – the only one
  that has not yet been solved.
- Arises from attempts to understand
  the pattern of the primes.

Prime numbers

- N is a prime number if its only divisors are 1 and N.
- Primes less than 20:
- 2, 3, 5, 7, 11, 13, 17, 19
- Non-primes less than 20:
- 4, 6, 8, 9, 10, 12, 14, 15, 16, 18

The primes thin out

- There are 4 primes below 10
- There are 25 primes below 100
- There are 168 primes below 1,000
- Euclid (ca 350 B.C.): There are infinitely many primes.

The density of the primes

- Let P(N) = # primes below N
- Let D(N) = P(N)/N
- Table:

***N*:**	**10**	**100**	**1,000**	**10,000**	**100,000**	**1,000,000**
D(*N*):	**.4**	**.25**	**.168**	**.123**	**.096**	**.078**

- Question: Is there a pattern?

Gauss' Conjecture

- 1791: Karl Friedrich Gauss conjectured that the bigger N gets, the closer D(N) gets to the function 1/ln(N).
- He could not prove this.
- 1896: Finally proved by Jacques Hadamard and Charles de la Vallée Poussin. Called the **Prime Number Theorem**.

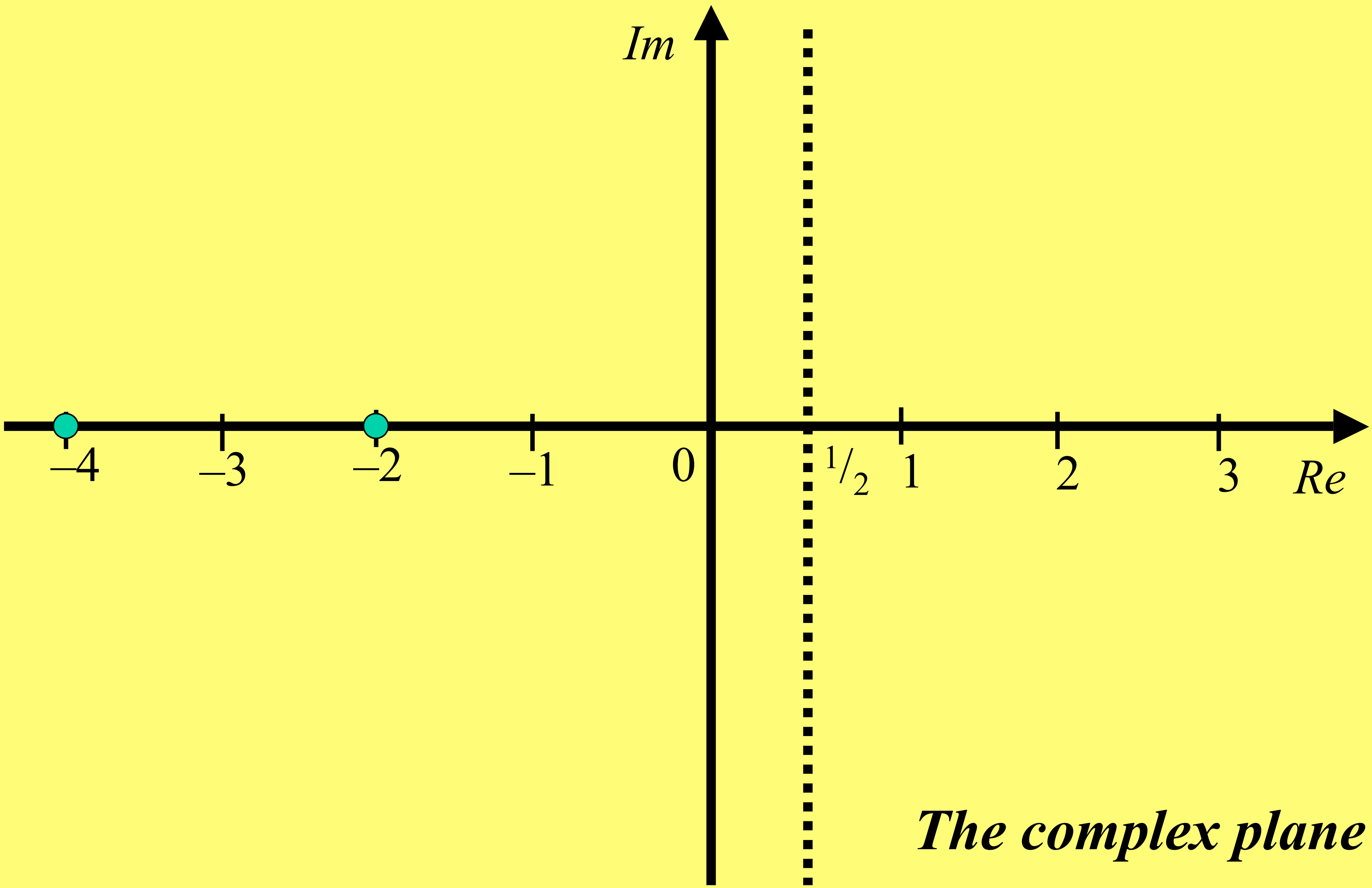
The Riemann zeta function

The proof of the Prime Number Theorem depended on some work of Riemann, who showed that information about the density function D(N) could be obtained from the solutions to the equation $\zeta(s) = 0$, for a particular function $\zeta(s)$ of complex numbers s.

The Riemann zeta function

- Riemann knew that $s = -2, -4, -6, \ldots$ are all solutions, and that there are infinitely many complex solutions.
- He conjectured that all the others were of the form $^1/_2 + ir$ for real numbers r.

Riemann's conjecture



The complex plane

The Riemann Hypothesis

All nonreal solutions to the equation

$$\zeta(s) = 0$$

are of the form $^1/_2 + ir$

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Using computers, the conjecture has been verified for the first 1.5 billion solutions (i.e., the 1.5 billion closest to the real axis).

The Riemann Hypothesis

All nonreal solutions to the equation

$$\zeta(s) = 0$$

are of the form $^1/_2 + ir$

Using computers, the conjecture has been verified for the first 1.5 billion solutions (i.e., the 1.5 billion closest to the real axis). *But is it true?*

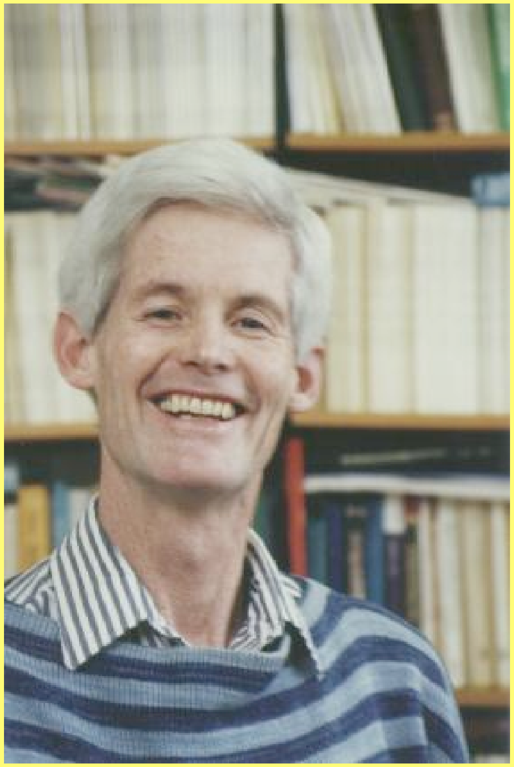
The Navier-Stokes Equations

Solve the partial differential equations that explain how a fluid flows when subjected to various forces.

The equations were formulated by Claude Henri Navier and George Stokes in the early 19th Century.

The P = NP Problem

If a computational problem is such that you can check any proposed answer in polynomial time, can you solve the problem in polynomial time?



Posed by Stephen Cook (and independently by Leonid Levin) in the early 1970s.

The Yang-Mills Equations & the Mass Gap Hypothesis

The Yang-Mills equations describe the fundamental forces of nature. Solve them and show that the solution implies that particles have a minimum mass.

The equations were formulated by the physicists Chen-Ning Yang and Robert Mills in the 1950s.

The Conjecture of Birch and Swinnerton-Dyer

There are infinitely many rational points on the elliptic curve E if and only if $\zeta_E(1) = 0$.

Formulated in the early 1960s by Brian Birch & Peter Swinnerton-Dyer

The Hodge Conjecture

Formulated by the British mathematician Sir William Vallance Douglas Hodge in 1950.

It's a problem about algebraic varieties (generalized forms of geometric figures).

It says that certain constructions that seem to require calculus really don't.

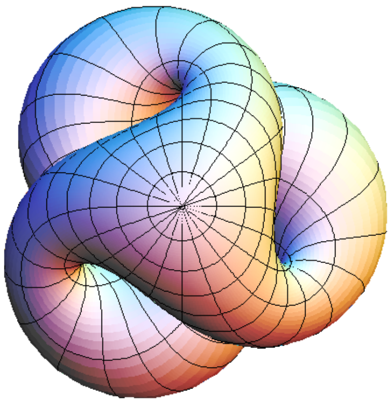
The Poincaré Conjecture

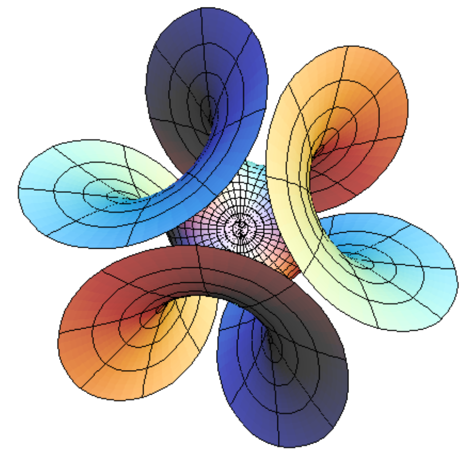
A conjecture about the kinds of 3-dimensional analogues of surfaces there can be, raised by Henri Poincaré in 1904.

Henri Poincaré

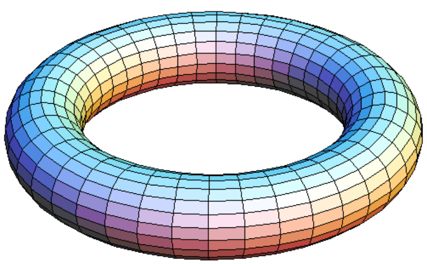
- Born and lived in France, 1854 – 1912
- Main interests physics and mathematics
- Was a renowned expositor of mathematics
- Published the book *Analysis situs* in 1895, the first complete treatment of topology. Beginning of algebraic topology
- Was led to topology by work in differential equations and multiple integrals.

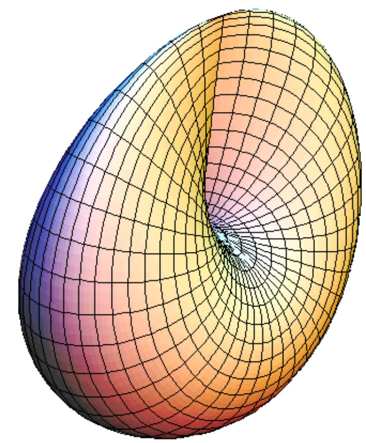
Topology



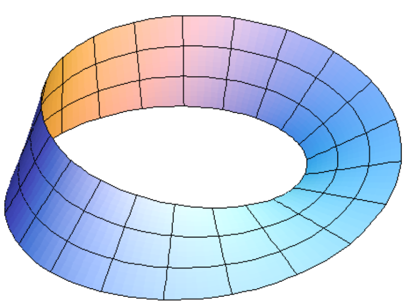


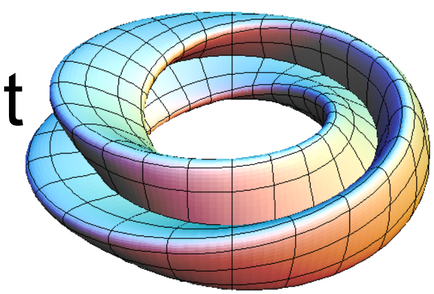
- *Topos logos* – the study of position
- Poincaré had a particular interest in manifolds





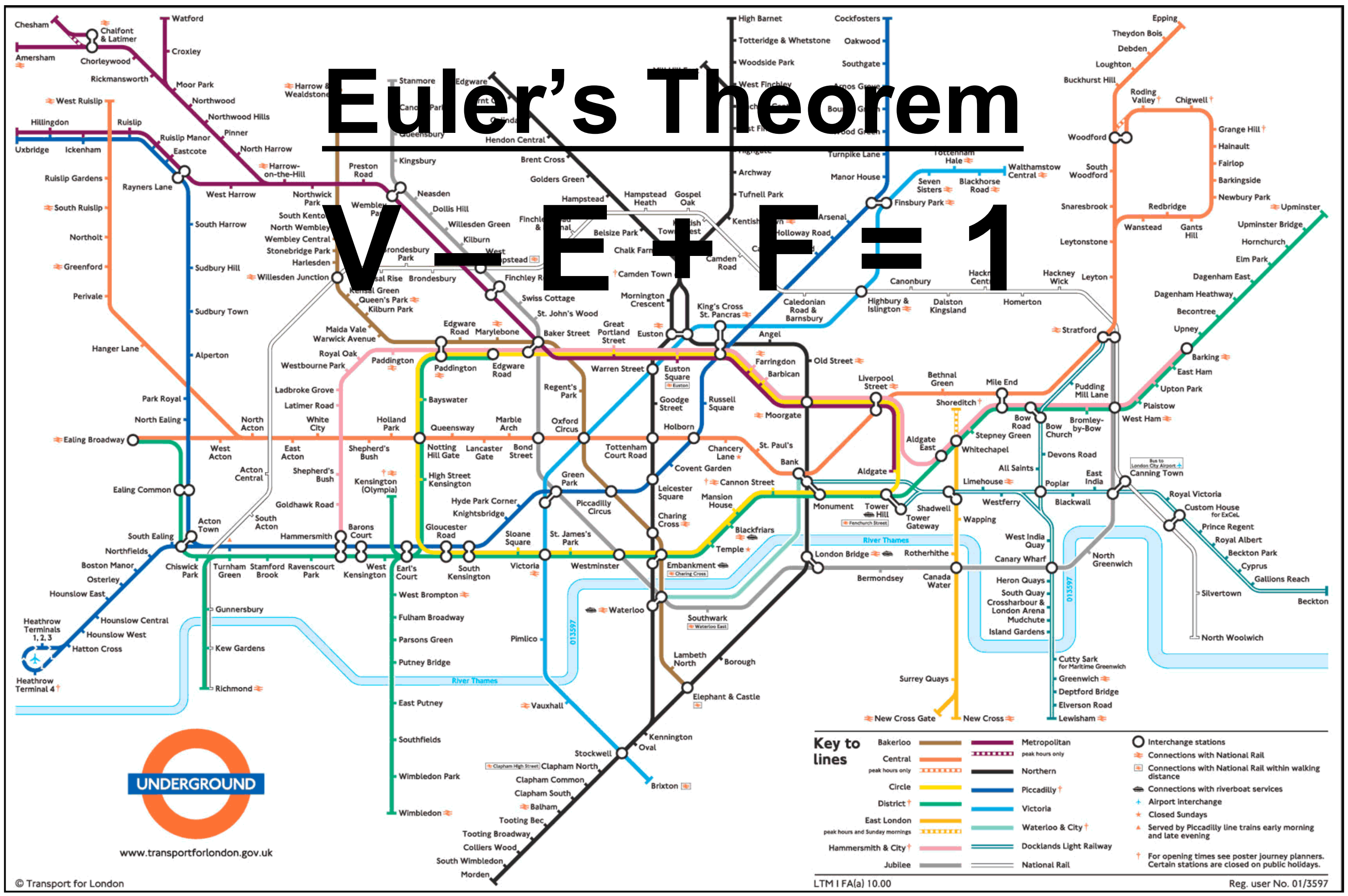
- 2-D manifolds are surfaces (two-dimensional objects in space of three or more dimensions)





- 3-D manifolds are important in modern physics

Topology of surfaces
(Rubber sheet geometry)



Euler's Theorem

$$V - E + F = 1$$

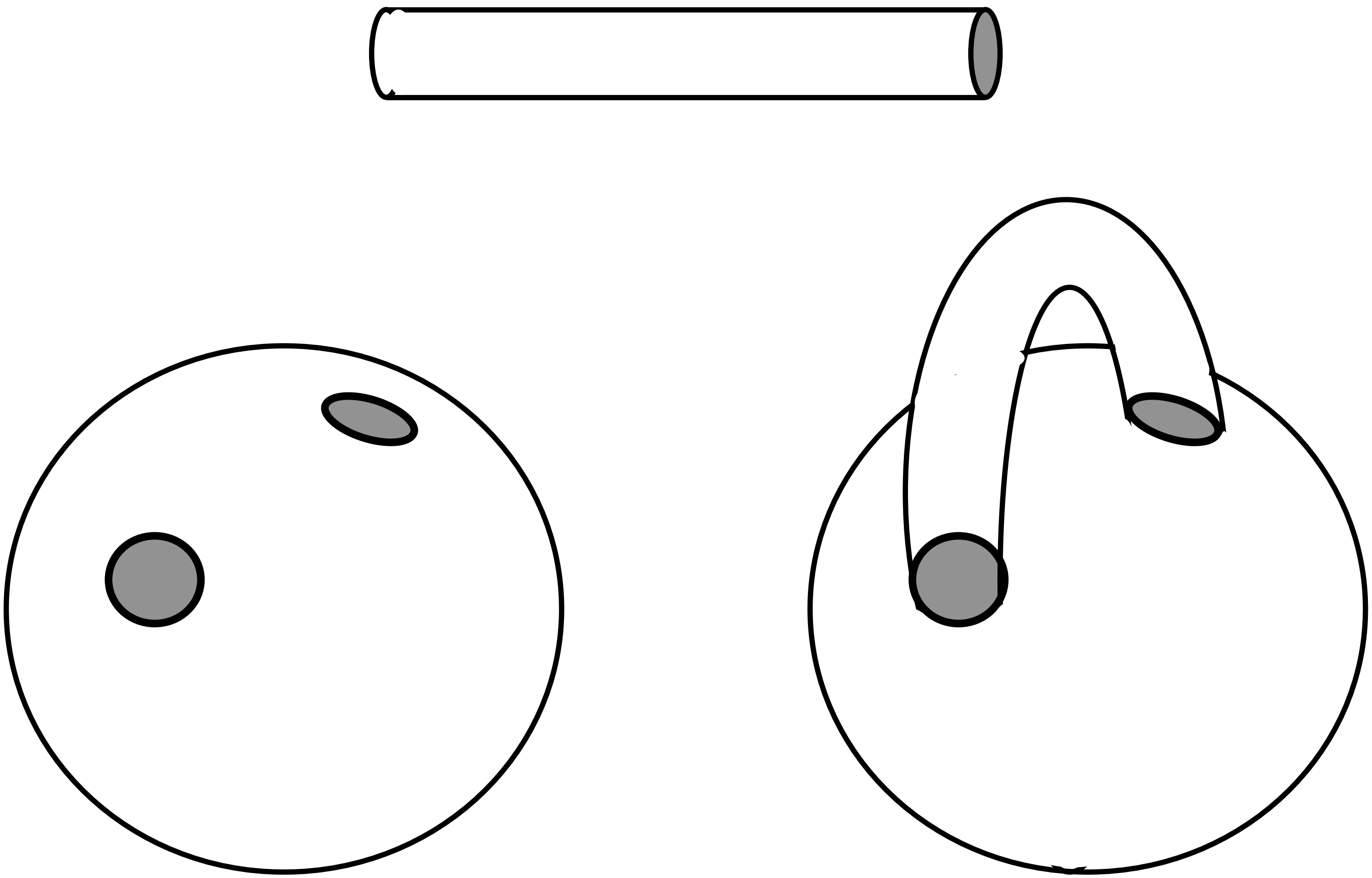
Classification of surfaces

<u>Riemann and others (19th Century)</u>: All smooth, closed surfaces are classified by two topological properties:

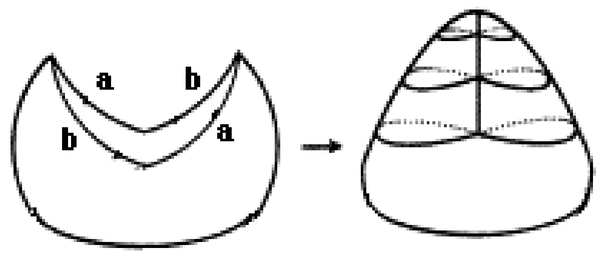
1. Orientability (handedness)
2. Euler characteristic (V-E+F)

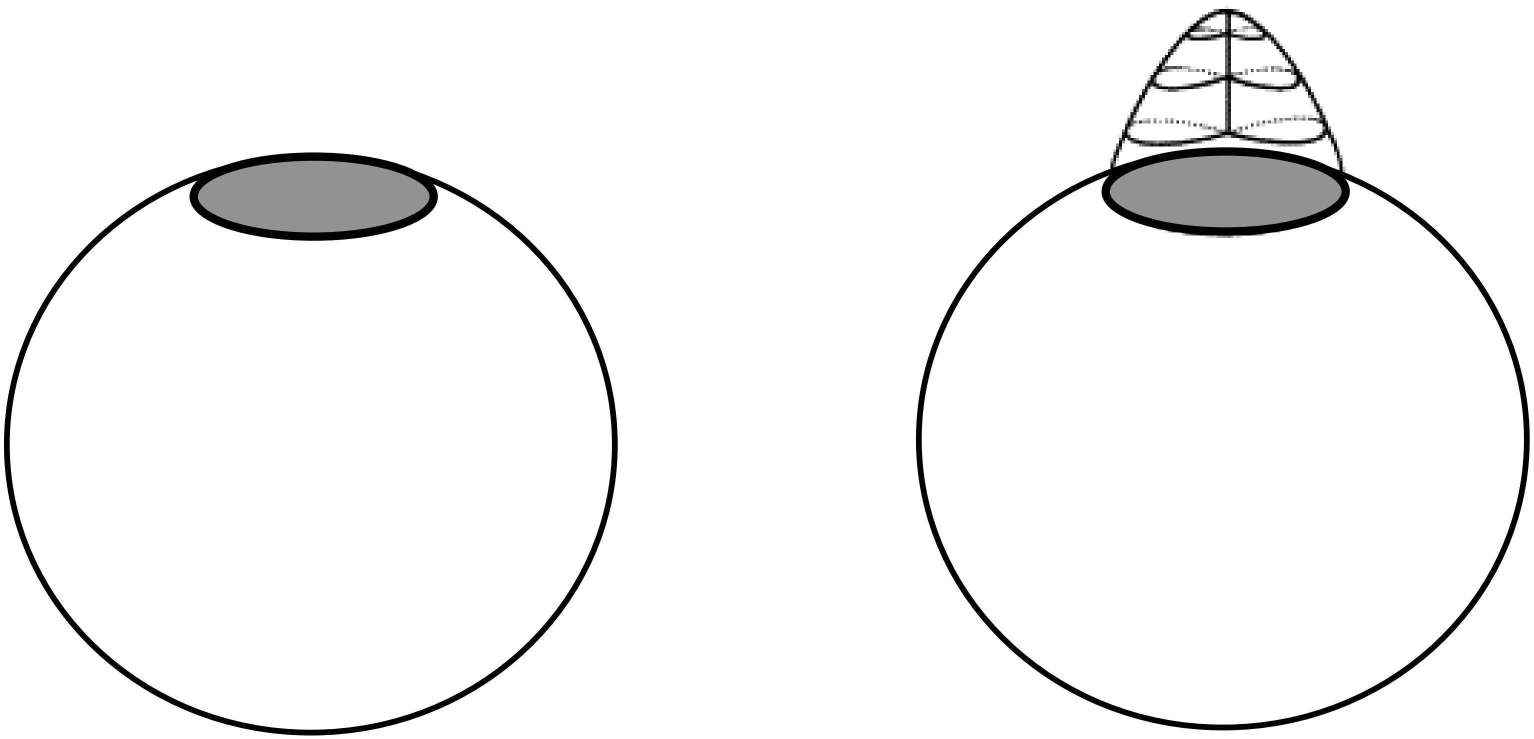
<u>Proof</u>: Take the sphere as basic and show how any smooth, closed surface can be obtained from it by means of “surgery”

Surgery: Adding a handle



Surgery: Adding a crosscap





Poincaré's idea for classifying all smooth, closed 3-manifolds

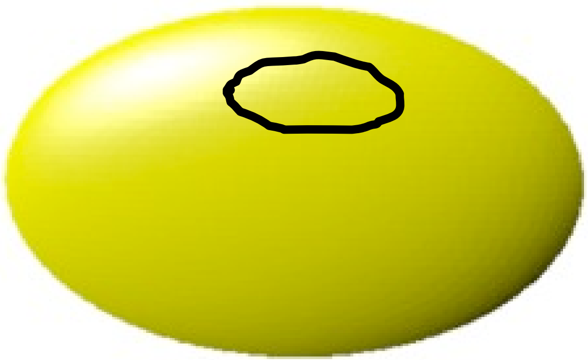
- Take the 3-sphere as basic and show how any smooth, closed 3-manifold can be obtained from it by means of surgery
- Need to start with a simple topological characterization of a 3-sphere. But what?
- Can we generalize something that works for the 2-sphere?
- Loop shrinking property (homotopy)?

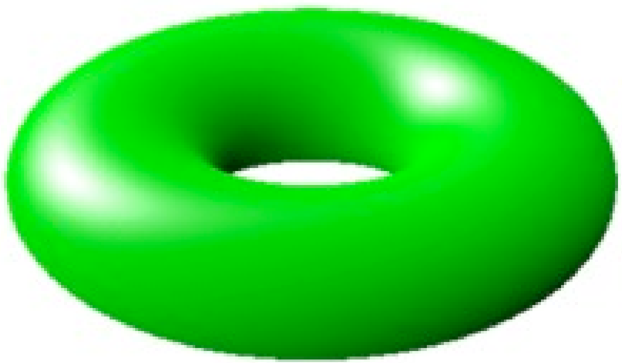
Characterization of a 2-sphere

Loop shrinking property

Characterization of a 2-sphere

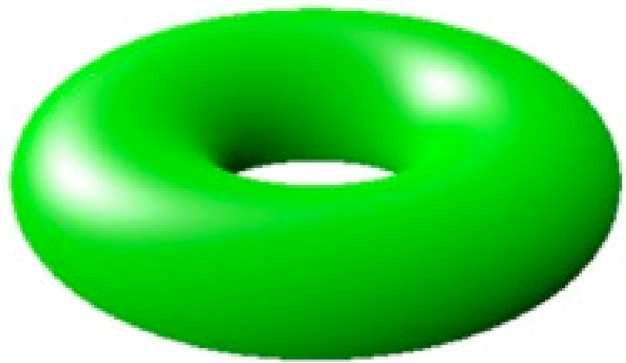
Loop shrinking property





Characterization of a 2-sphere

Loop shrinking property



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Poincaré's idea for classifying all smooth, closed 3-manifolds

- Take the 3-sphere as basic and show how any smooth, closed 3-manifold can be obtained from it by means of surgery
- Need to start with a simple topological characterization of a 3-sphere
- Loop shrinking property (homotopy)
- Works for 2-sphere
- Does it also work for 3-spheres?

Poincaré's idea for classifying all smooth, closed 3-manifolds

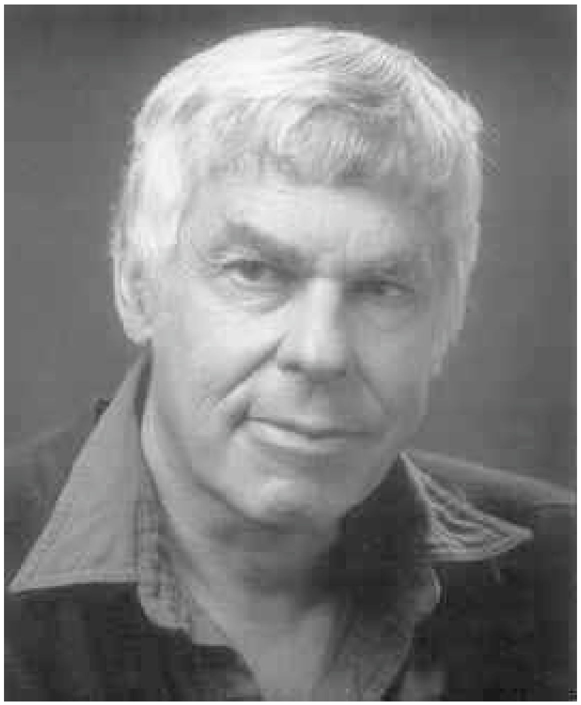
- Take the … w how any smo … d from it b…
- Nee… charact…
- Loop shrinking p… perty (homotopy)?
- Works for 2-sph…
- **Does it also work for 3-spheres?**

The Poincaré Conjecture says the answer is "Yes"

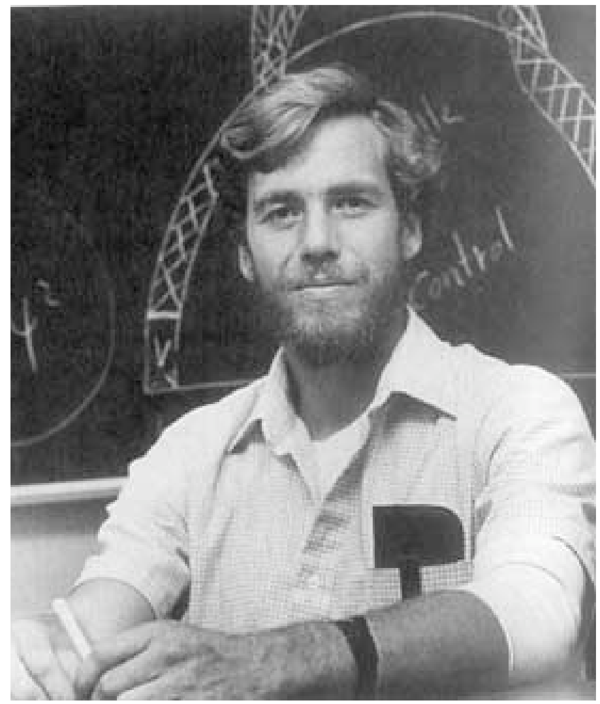
The Poincaré Conjecture

Prove that the only three-dimensional manifold in which every closed loop (or hyperloop) can be continuously deformed to a point is the 3-sphere.

The Generalized Poincaré Conjecture



1961: Stephen Smale proved it for all dimensions greater than 4



1982: Michael Freedman proved it for dimension 4

The (Original) Poincaré Conjecture

- Remained unsolved, and in 2000 was declared a $1m Millennium Problem
- It is still unsolved
- Or is it?