Stanford Continuing Studies, Course MATH 07, Fall 2005

Three views of mathematics

LECTURE 4

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The Millennium Problems

P versus NP

The Hodge Conjecture

The Poincaré Conjecture

The Riemann Hypothesis

Yang-Mills Existence and Mass Gap

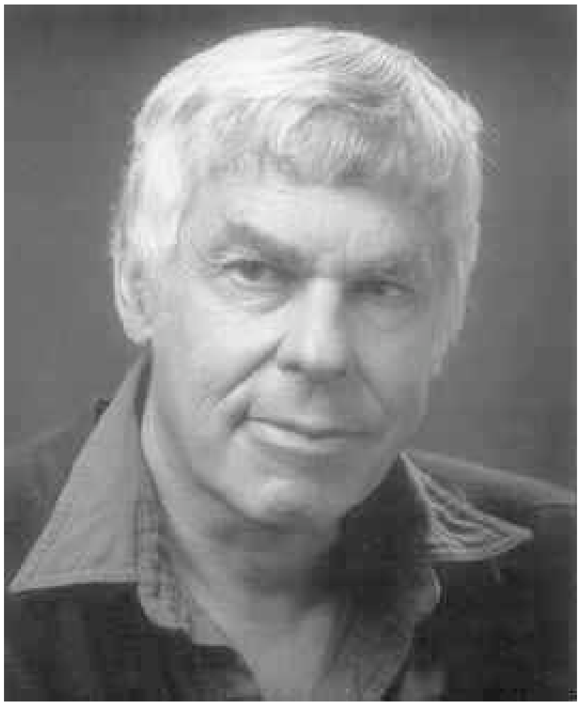
Navier-Stokes Existence and Smoothness

The Birch and Swinnerton–Dyer Conjecture

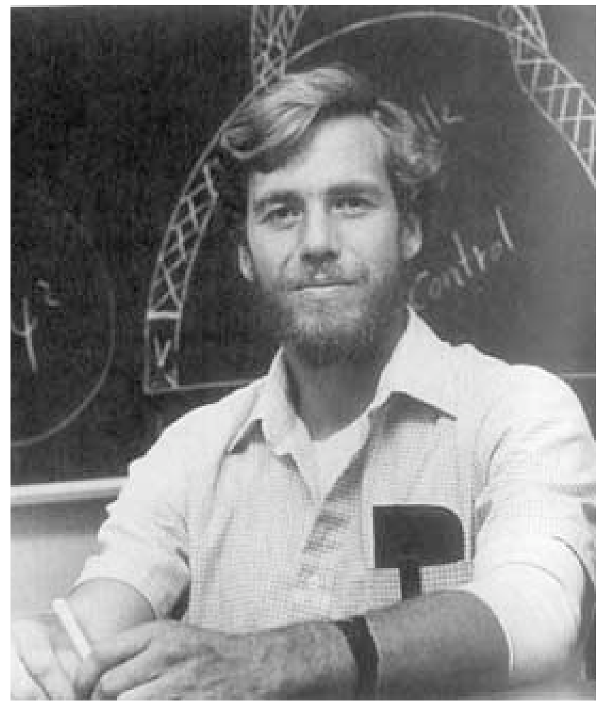
The Poincaré Conjecture

Prove that the only three-dimensional manifold in which every closed loop (or hyperloop) can be continuously deformed to a point is the 3-sphere.

The Generalized Poincaré Conjecture



1961: Stephen Smale proved it
for all dimensions greater than 4

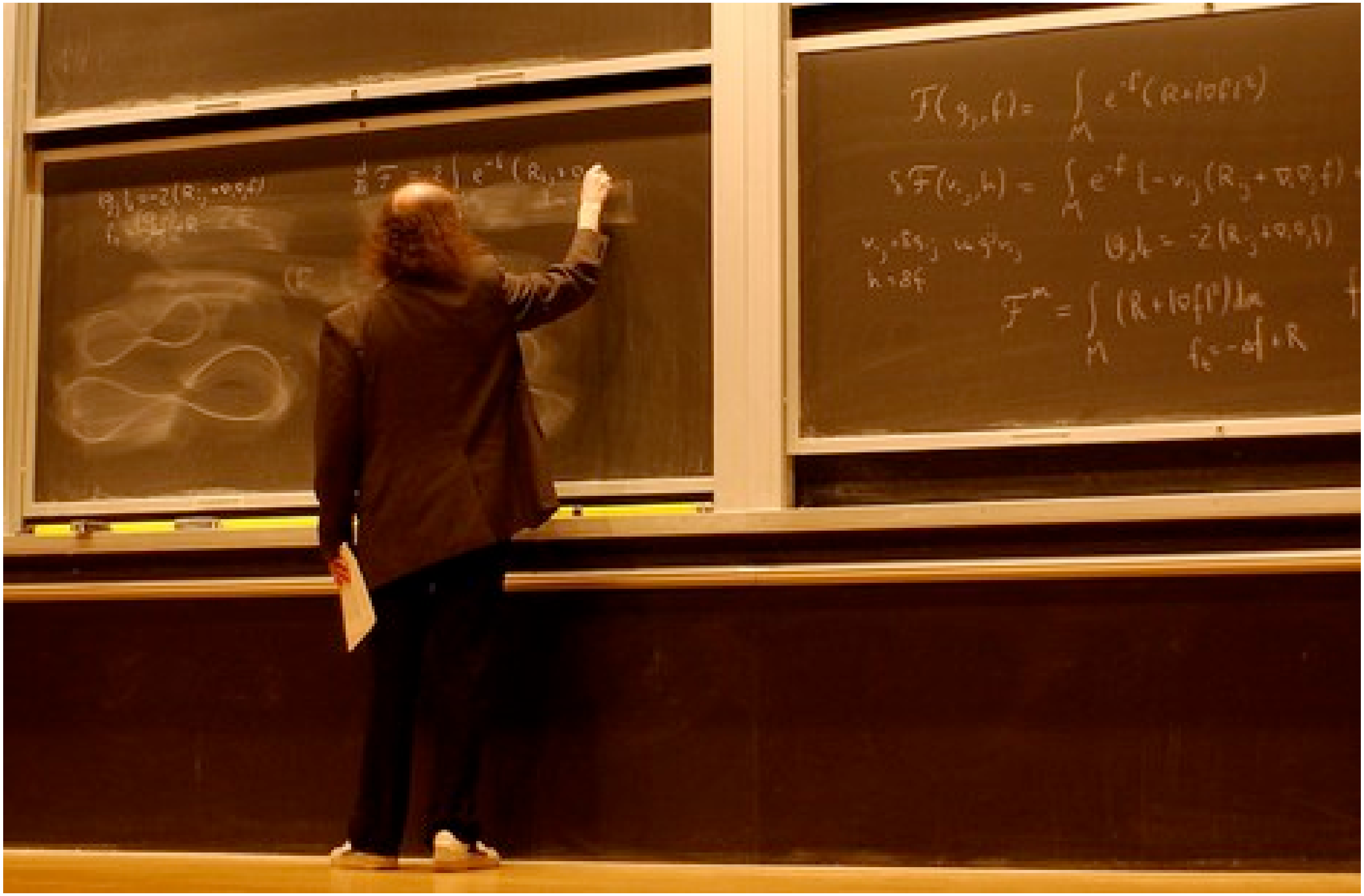


1982: Michael Freedman
proved it for dimension 4

The (Original) Poincaré Conjecture

- Remained unsolved, and in 2000 was declared a $1m Millennium Problem
- It is still unsolved
- Or is it?

Grigori (Grisha) Perelman



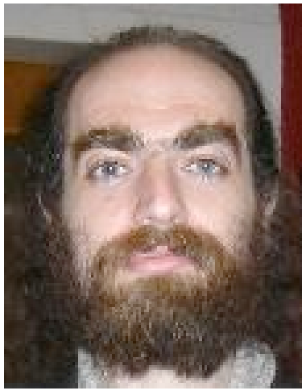
Mathematics, abstract
math.DG/0211159

From: Grisha Perelman [view email]
Date: Mon, 11 Nov 2002 16:11:49 GMT (33kb)

The entropy formula for the Ricci flow and its geometric applications

Authors: Grisha Perelman
Comments: 39 pages
Subj-class: Differential Geometry
MSC-class: 53C

We present a monotonic expression for the Ricci flow, valid in all dimensions and without curvature assumptions. It is interpreted as an entropy for a certain canonical ensemble. Several geometric applications are given. In particular, (1) Ricci flow, considered on the space of riemannian metrics modulo diffeomorphism and scaling, has no nontrivial periodic orbits (that is, other than fixed points); (2) In a region, where singularity is forming in finite time, the injectivity radius is controlled by the curvature; (3) Ricci flow can not quickly turn an almost euclidean region into a very curved one, no matter what happens far away. We also verify several assertions related to Richard Hamilton's program for the proof of Thurston geometrization conjecture for closed three-manifolds, and give a sketch of an eclectic proof of this conjecture, making use of earlier results on collapsing with local lower curvature bound.



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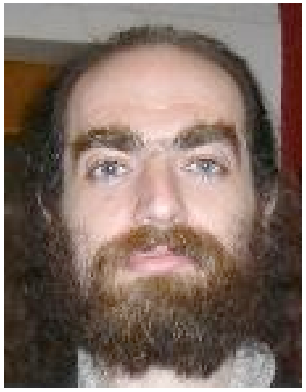

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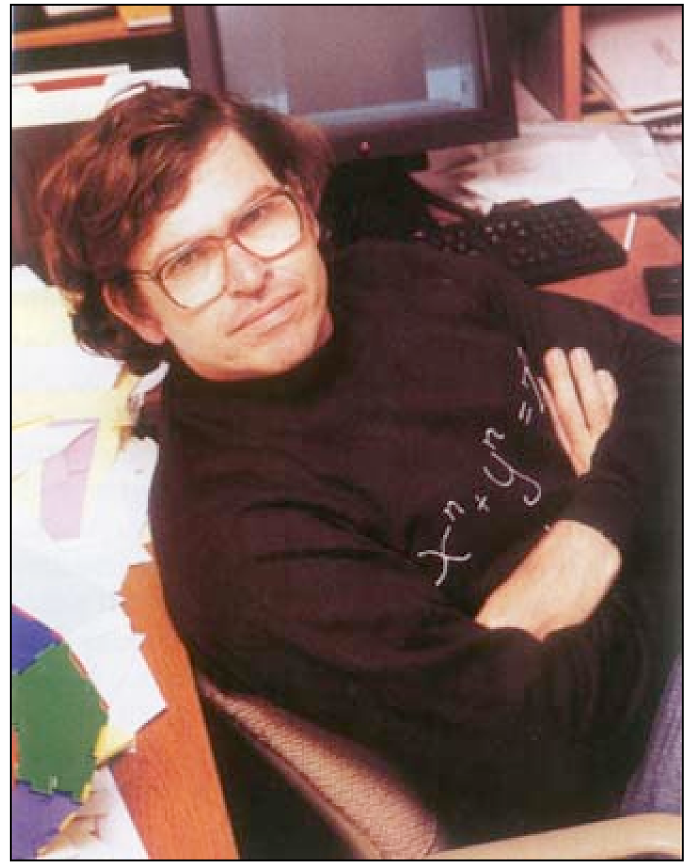
The Geometrization Conjecture

Proposed by William Thurston in the late 1970s.

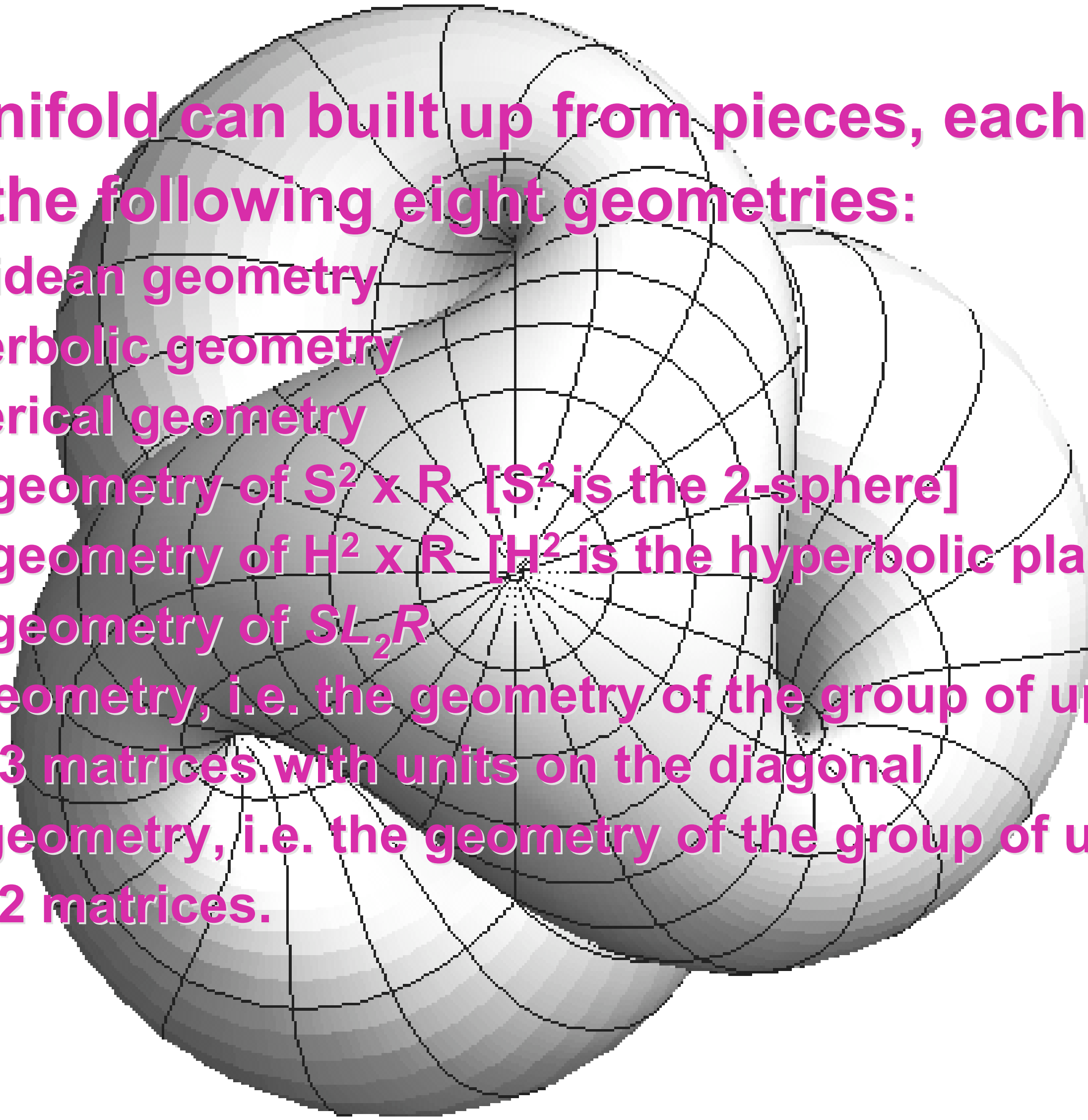
It implies the Poincaré Conjecture as a special case.



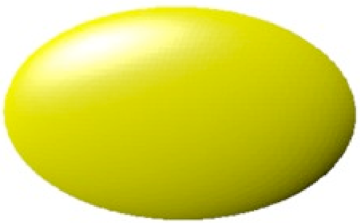
The Geometrization Conjecture

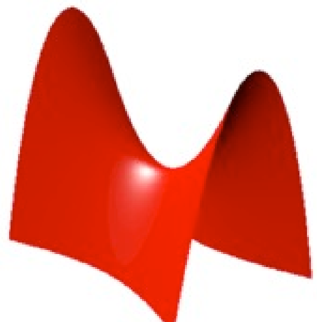
Every 3-manifold can built up from pieces, each of which has one of the following eight geometries:

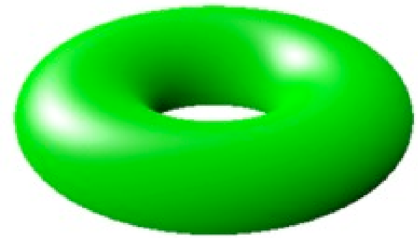
1. **Euclidean geometry**
2. **Hyperbolic geometry**
3. **Spherical geometry**
4. **The geometry of S^2 x R [S^2 is the 2-sphere]**
5. **The geometry of H^2 x R [H^2 is the hyperbolic plane]**
6. **The geometry of SL_2R**
7. **Nil geometry, i.e. the geometry of the group of upper triangular 3 by 3 matrices with units on the diagonal**
8. **Sol geometry, i.e. the geometry of the group of upper triangular 2 by 2 matrices.**



The Geometrization Conjecture alternative formulation







Every irreducible, compact 3-manifold falls into exactly one of the following categories:

1. **It has a spherical geometry**
2. **It has a hyperbolic geometry**
3. **The fundamental group contains a subgroup isomorphic to the free abelian group on two generators (the fundamental group of a torus).**

Perelman's Approach

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- General agreement that he can handle problem 1.
- Less consensus he has tamed problem 2.

Current Status
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Poincaré Conjecture

Maybe solved

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- Formal definition: a proof of X is a finite sequence $X_1,\ldots,X_n$ of statements such that $X_n = X$ and each X_i is either an axiom or else follows from $X_1,\ldots X_{i-1}$ by a single application of a recognized rule of logical deduction.

Example of a logical rule

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From P and $P \rightarrow Q$, deduce Q

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Example:

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P $\rightarrow$ Q: If Julie captain the team, the team wins.

Modus ponens: “If you know that the team wins whenever Julie captains it, then if you know that Julie is captaining the team, you can conclude that the team wins.”

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Look at the number $Q = P_1 \times \ldots \times P_N + 1$.

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If Q is not prime, it is divisible by some prime number P less than Q.

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P cannot be any of P_1, P_2, …, P_N, since dividing any of those into Q leaves a remainder of 1. Hence P is a different prime number, which must be bigger than P_N.

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Either way, there is a prime number bigger than P_N.

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- The four color theorem: Prove that any map drawn in the plane may be colored using at most four colors.
- Classification of finite simple groups: Describe the different kinds of finite simple groups.
- The Kepler Conjecture: Prove that the face-centered cubic lattice (the orange-pile configuration) is the most efficient way to pack identical spheres in space.

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