Stanford Continuing Studies, Course MATH 07, Fall 2005

# Three views of mathematics

## LECTURE 5

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# Challenge Problems

Sometimes restrict the kinds of solution that are acceptable.

- Ruler-and-compass constructions
  - eg. Squaring the circle
- Whole number solutions (Diophantine equations)
  - eg. Fermat's Last Theorem
- Closed form solutions.
  - eg. Navier-Stokes equations Millennium Problem

# NOTE: Closed form solutions

MATHWORLD definition: “An equation is said to be a closed-form solution if it solves a given problem in terms of functions and mathematical operations from a given generally accepted set.  For example, an infinite sum would generally not be considered closed-form.  However, the choice of what to call closed-form and what not is rather arbitrary since a new "closed-form" function could simply be defined in terms of the infinite sum.”

# The nature of mathematical proof

- Intuitive idea: a proof of X is a piece of reasoning (an argument) that convinces a suitably qualified expert that X is true. [SOCIOLOGICAL]

- Formal definition: a proof of X is a finite sequence $X_1,\ldots,X_n$ of statements such that $X_n = X$ and each $X_i$ is either an axiom or else follows from $X_1,\ldots X_{i-1}$ by a single application of a recognized rule of logical deduction. [IDEALIZED, FORMAL]

# The axiomatic method and its limits

Gödel's (First) Incompleteness Theorem (1931): If $\mathcal{A}$ is a consistent set of axioms in a formal language $\mathcal{L}$ that is sufficiently strong to yield elementary arithmetic, then there is a sentence S in the language $\mathcal{L}$ which is true but which cannot be formally deduced from $\mathcal{A}$.

Gödel's Second Incompleteness Theorem: One such sentence S is the statement that $\mathcal{A}$ is consistent.

# The axiomatic method and its limits

Idea for the proof of the (First) Incompleteness Theorem: Formulate a sentence S of arithmetic that says, effectively, “S is not provable.”

cf. The Liar Paradox, where a person says “I am lying.” (More fully: “The sentence I am now uttering is false.”)