Stanford Continuing Studies, Course MATH 07, Fall 2005

Three views of mathematics

LECTURE 6

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Mathematical probability

Let A be some action (such as tossing a coin, rolling a die, or spinning a roulette wheel), and let E be some specified outcome (such as getting a head, rolling a 6, or landing on black). The **probability** of E is defined to be the ratio

$$P_A(E) = \frac{\text{number of ways E can occur}}{\text{number of possible outcomes of A}}.$$

Mathematical probability

For coin tossing, the possible outcomes are H and T. The probability of getting a head is

$$P_A(H) = \frac{1}{2} = 0.5$$

For rolling a die, the possible outcomes are 1, 2, 3, 4, 5, 6. The probability of getting an even number is

$$P_A(E) = \frac{3}{6} = 0.5$$

Mathematical probability

If two die are rolled, what is the probability that they will sum to 6?

There are 36 possible outcomes:

1 - 1	1 - 2	1 - 3	1 - 4	1 - 5	1 - 6
2 - 1	2 - 2	2 - 3	2 - 4	2 - 5	2 - 6
3 - 1	3 - 2	3 - 3	3 - 4	3 - 5	3 - 6
4 - 1	4 - 2	4 - 3	4 - 4	4 - 5	4 - 6
5 - 1	5 - 2	5 - 3	5 - 4	5 - 5	5 - 6
6 - 1	6 - 2	6 - 3	6 - 4	6 - 5	6 - 6

Mathematical probability

If two die are rolled, what is the probability that they will sum to 6?

Which outcomes sum to 6?

1 - 1	1 - 2	1 - 3	1 - 4	1 - 5	1 - 6
2 - 1	2 - 2	2 - 3	2 - 4	2 - 5	2 - 6
3 - 1	3 - 2	3 - 3	3 - 4	3 - 5	3 - 6
4 - 1	4 - 2	4 - 3	4 - 4	4 - 5	4 - 6
5 - 1	5 - 2	5 - 3	5 - 4	5 - 5	5 - 6
6 - 1	6 - 2	6 - 3	6 - 4	6 - 5	6 - 6

$$P(sum=6) = 5/36 = 0.1389$$

Figuring the odds

- Suppose that when a married couple has a child, there is an equal chance of the baby being male or female.
- What is the probability that a couple with two children will have two boys?
- A boy and a girl? Two girls?
- Answers: 1/4, 1/2, and 1/4.

Figuring the odds

- Four possibilities (in order of birth in each case): B-B, B-G, G-B, G-G.
- Each of these is equally likely.
- So there is a 1-in-4 probability that the couple will have two boys,
- a 2-in-4 (i.e., 1/2) probability of having one of each gender,
- a 1-in-4 probability of having two girls.

Assessing your knowledge

- You learn that I have a daughter who works at Google, and I tell you I have two children.
- What would you think is the probability that I have two girls?
- Most common answer is 1/2.
- Based on what you know, the probability that I have two girls is 1/3.

Assessing your knowledge

- Listed in birth order, the possible combinations are: B-B, B-G, G-B, G-G.
- You know that I have (at least) one daughter, which eliminates B-B.
- That leaves: B-G, G-B, G-G.
- Of these three, only one is G-G.
- Thus, the probability you compute that I have two girls is 1/3.

What is probability?

- Frequentist definition.
  - An empirical notion.
  - Works when there is an action, having a fixed number of possible outcomes, that can be repeated indefinitely.
- Subjective probability.
  - A numerical assessment of your knowledge of some event.
  - It quantifies **your knowledge of the event**, not the event itself.
  - Agrees with the frequentist notion when that applies.
  - Different people can assign different probabilities to their individual knowledge of the same event.
  - The probability you assign to an event depends on your prior knowledge of the event, and can change when you acquire new information about it.

What is probability?

- Frequentist probability doesn’t make much sense for actions that cannot be repeated indefinitely.
- We can regard frequentist probability as the subjective probability of a future performance of the action. For example, when you say that the probability of rolling a double-six with a roll of two dice is 1/36, you are quantifying your knowledge of how the roll will turn out.
- People often confuse subjective and frequentist probability.

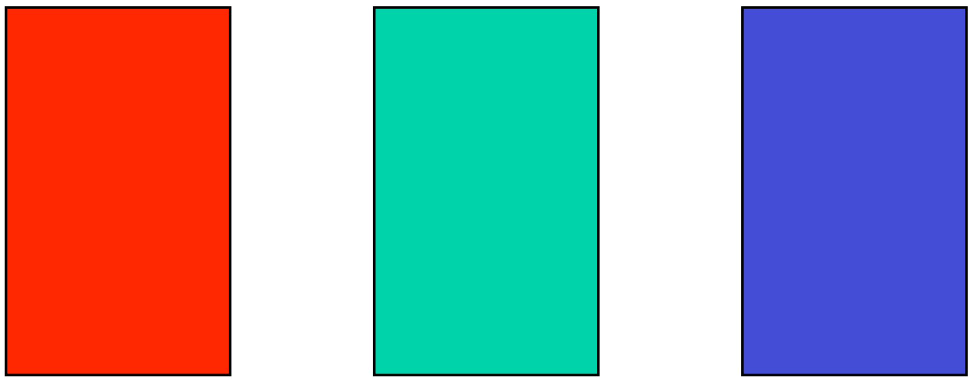
The Monty Hall Problem

- *Let's make a deal*.
- Three doors.
- The $10,000 prize is behind one door.
- You choose one door.
- Monty opens one of the doors you did not pick, to reveal no prize. (He knows where the prize is, and always chooses a door that does not hide the prize.)
- Monty asks you if you want to switch doors or keep your original choice?

The Monty Hall Problem

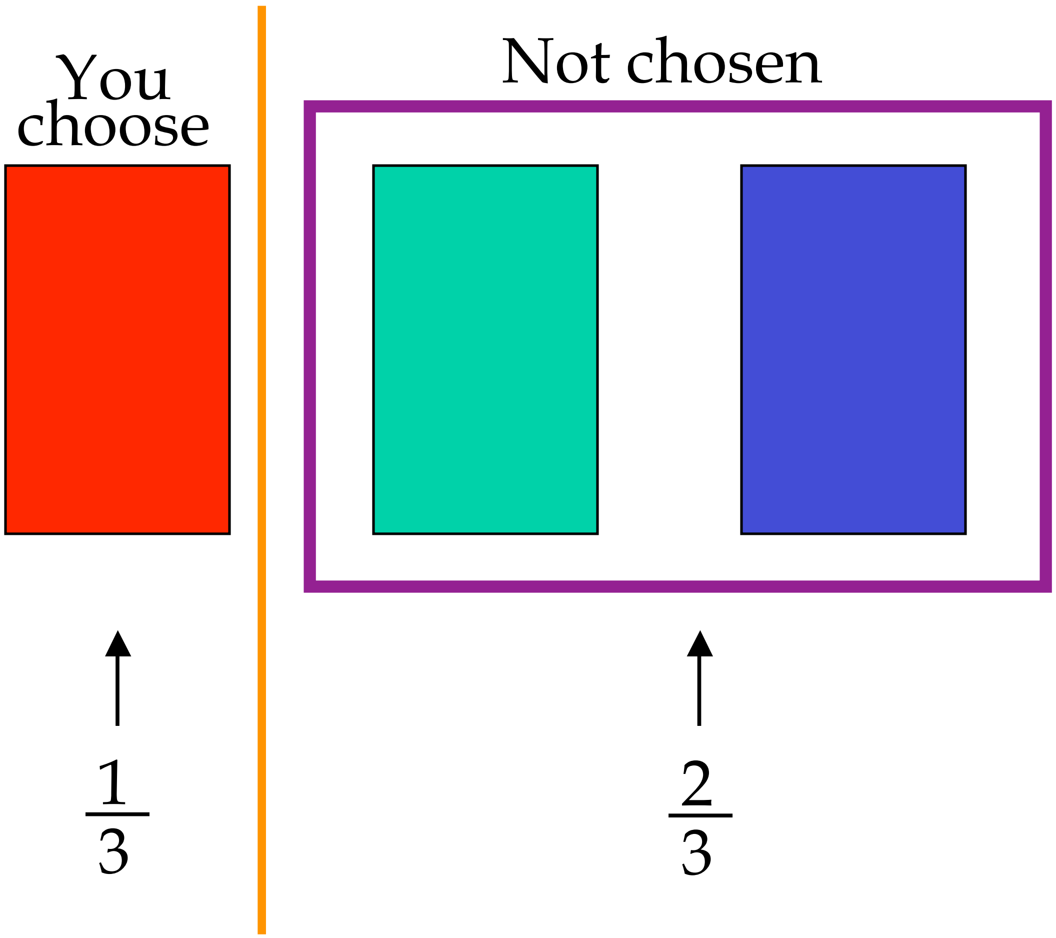
- Question: Do you switch or stick?
- Common answer: It makes no difference.
- Correct answer: You should switch.
- Changing your choice *doubles* your likelihood of winning.
- The probability of you winning goes up from 1-in-3 to 2-in-3.

The Monty Hall Problem

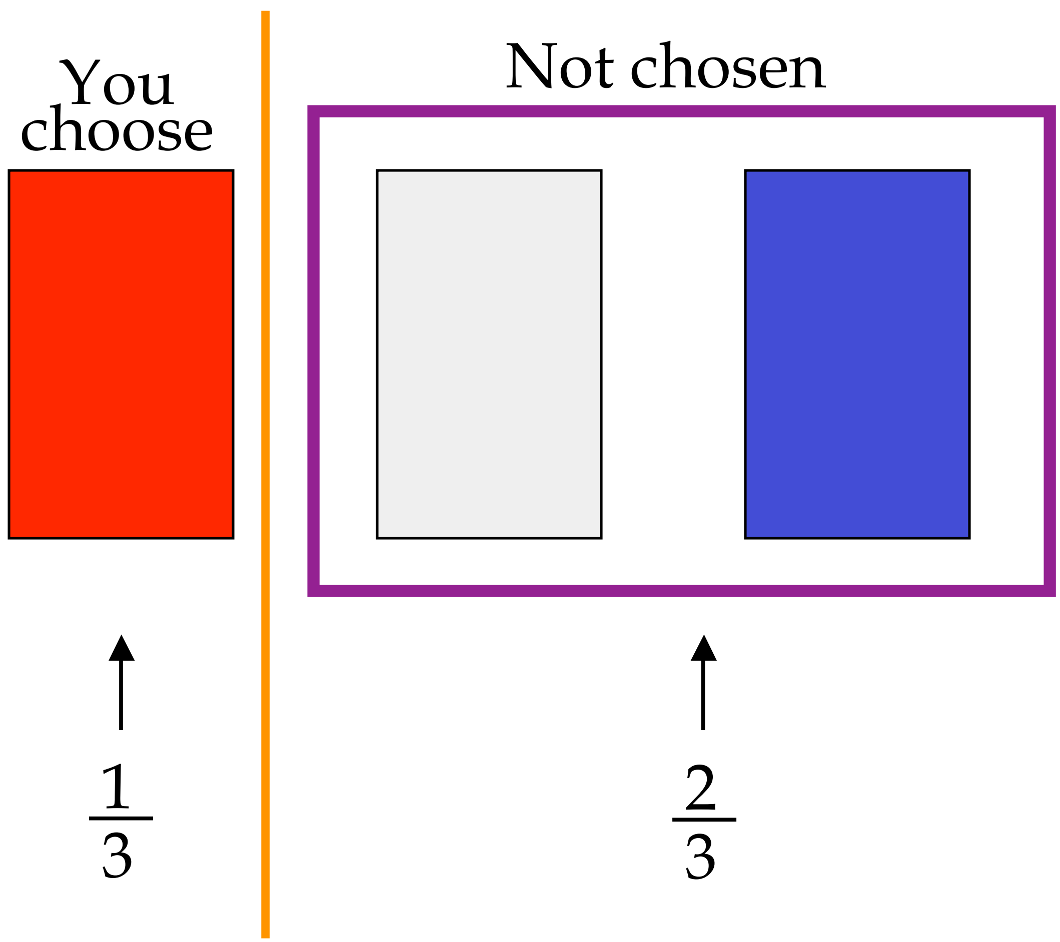


$$\frac{1}{3} \qquad \frac{1}{3} \qquad \frac{1}{3}$$

The Monty Hall Problem



The Monty Hall Problem



The arithmetic of probabilities

- $0 \leq P(E) \leq 1$
- P(E) = 0 if and only if E is impossible
- P(E) = 1 if and only if E is certain
- P(not-E) = 1 – P(E)
- If two events E and F are independent of each other (i.e., one does not affect the other), then P(E&F) = P(E) x P(F)
  - For example, the probability that when you throw two dice, the first comes up even and the second lands on 5, is 1/2 x 1/6 = 1/12
- If two events E and F are independent of each other, then P(E or F) = 1 – P(not-E) x P(not-F)
  - For example, the probability that when you throw two dice at least one of them is even, is 1 – P(odd) x P(odd) = 1 – .5 x .5 = .75

Birthday coincidences

- How many people do you need to join you in a room for there to be a better-than-evens chance of someone sharing your birthday?
- Common answer: 182.
- Correct answer: 253.
- If two events E and F are independent of each other, then P(E or F) = 1 – P(not-E) x P(not-F)

Birthday coincidences

- The probability of any person in the room having a different birthday from you is 364/365.
- With n people in the room, the probability of them all having a different birthday from you is $(364/365)^n$.
- The first n for which $(364/365)^n$ drops below 0.5 is $n = 253$.

Another birthday puzzle

- At a soccer match, there are 22 players and one referee on the field. What is the probability that at least two of them share the same birthday?
- Most people think the probability is quite low.
- In fact it’s 0.508, better than evens.

Another birthday puzzle

- With one person on the field, the probability of no shared birthdays is 1.
- With a second person, there are 364 possible days for that new person to have a birthday that is not the same as the person already on the field. So the probability of the two of them having different birthdays is 1 x 364/365.

Another birthday puzzle

- With a third person, there are 363 days on which that person's birthday can fall so all three have different birthdays.
- So the probability of all three birthdays being different is 1 x 364/365 x 363/365.
- With four people, the probability of having no shared birthday is 1 x 364/365 x 363/365 x 362/365.

Another birthday puzzle

- When all 23 are on the field, the probability that all birthdays are different is:

  1 x 364/365 x 363/365 x 362/365 x ... x 343/365.
- This works out to be 0.492.
- So, the probability that at least two people share the same birthday is 1 – 0.492 = 0.508.

A Moral

- Puzzles such as the Monty Hall challenge and the birthday coincidence example cause us great difficulty. Why?
- Evolution has not equipped us to assess risks in “complicated” situations.
- We tend to base our estimates on the most recent or most immediate piece of evidence and ignore surrounding or prior circumstances.
- If we want to make a reliable estimate, we have to use mathematics.

The cancer test

- The cancer has an incidence of 1%.
- The reliability of the test is 79%.

  i.e., it detects the cancer when present, but has a 21% rate of false positives.
- Your test shows positive.
- What is the probability that you have the cancer?
- Answer: 4.6%.

Solution to the cancer test problem

- Assume a population of 10,000 people.
- Assume the various probabilities are reflected exactly in the actual numbers.
- Thus, of the total population of 10,000, 100 will have the cancer, 9,900 will not.
- In the absence of the test, the likelihood of you having the cancer is 1%.
- How do you revise that figure after the test?

Solution to the cancer test problem

- There are 100 individuals in the population who do have the cancer.
- For all of them the test will correctly give a positive prediction.
- Of the 9,900 cancer-free individuals, 21% will test positively.
- # of false positives: 9900 x 0.21 = 2079.
- Total # of positives: 100 + 2079 = 2179.

Solution to the cancer test problem

- # of false positives: 9900 x 0.21 = 2079.
- Total # of positives: 100 + 2079 = 2179.
- You are among the 2179 who test positive.
- Are you in the subgroup of 100 that really does have the cancer, or is your test result a false positive?
- The probability of you being among the true positives is 100/2179 = 0.046.
- In other words, there is a 4.6% possibility that you have the cancer.

Bayes' theorem

Thomas Bayes (1702-1761)

Bayes' theorem shows you to calculate the probability of a certain hypothesis H, based on evidence E, when you know:

(1) the probability of H in the absence of any evidence;

(2) the evidence (E) for H;

(3) the probability the evidence is correct.

Bayes' theorem

P(H|E) = conditional probability of H given E.

$$P(H|E) = \frac{P(H) \times P(E|H)}{P(H) \times P(E|H) + P(H\text{-wrong}) \times P(E|H\text{-wrong})}$$

The cancer test

H = hypothesis that you have cancer.

E = the test is positive.

$$P(H|E) = \frac{P(H) \times P(E|H)}{P(H) \times P(E|H) + P(H\text{-wrong}) \times P(E|H\text{-wrong})}$$

$$= \frac{0.01 \times 1}{0.01 \times 1 + 0.99 \times 0.21}$$

$$= 0.046$$

Eyewitness testimony

- Hit-and-run accident.
- Two companies: Blue Cabs and Black Cabs.
- Blue Cabs has 15 taxis, Black Cabs has 85.
- Eyewitness: “It was a blue taxi.”
- In tests, the witness correctly identifies the color of the taxi 4 times out of 5.
- How likely is it that it was a Blue Cab?

Eyewitness testimony

- Common answer: Since the witness has shown he is right 4 times out of 5, you conclude that it was, as he said, a blue taxi.
- You might even think that the odds in favor of it being a blue taxi were exactly 4 out of 5 (i.e., a probability of 0.8).
- Correct answer: the probability it was a blue taxi is only 0.41. (Less than half.)

Eyewitness testimony

P(blue|witness) =

$$\frac{P(\text{blue}) \times P(\text{right})}{P(\text{blue}) \times P(\text{right}) + P(\text{black}) \times P(\text{wrong})}$$

$$= \frac{0.15 \times 0.8}{0.15 \times 0.8 + 0.85 \times 0.2}$$

$$= 0.12/[0.12 + 0.17] = 0.12/0.29 = 0.41$$

More on subjective probability

What does it mean when a weather person says that there is a 60% chance of rain tomorrow?

Or when say that you are 90% certain you turned the gas off before you left home?

Are these the same?

Can we make them precise?

Bruno de Finetti (1906-85)

The de Finetti game:

You say you are 90% certain you turned the gas off. I offer you a deal. I present you with a jar containing 100 balls, 99 of them red, 1 black.

You have a choice. Either you draw one ball from the jar, and if it's red, you win $1m. Or we can go back and check if the gas is on, and if it is not, I give you $1m.

Presumably, you will elect to pick a ball from the jar.

Hence, your **rational confidence** that you have turned off the gas is at most 99%.

Bruno de Finetti (1906-85)

Now I offer you a jar that contains 95 red balls and 5 black, with choosing a red ball netting you $1m.

Assuming you again choose to select a ball, your rational confidence that you have turned off the gas is at most 95%.

Then I offer a jar with 90 red balls and 10 black.

If you choose to pick a ball, your rational confidence that you have turned off the gas is at most 90%.

And so on.

Eventually, you decide you would prefer to check the gas. If that happens when there are N red balls in the jar, then your rational confidence is N%.